

# On the lower bound of the discrepancy of $(t, s)$ sequences: II

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## Abstract

Let  $(\mathbf{x}(n))_{n \geq 1}$  be an  $s$ -dimensional Niederreiter-Xing's sequence in base  $b$ . Let  $D((\mathbf{x}(n))_{n=1}^N)$  be the discrepancy of the sequence  $(\mathbf{x}(n))_{n=1}^N$ . It is known that  $ND((\mathbf{x}(n))_{n=1}^N) = O(\ln^s N)$  as  $N \rightarrow \infty$ . In this paper, we prove that this estimate is exact. Namely, there exists a constant  $K > 0$ , such that

$$\inf_{\mathbf{w} \in [0,1]^s} \sup_{1 \leq N \leq b^m} ND((\mathbf{x}(n) \oplus \mathbf{w})_{n=1}^N) \geq Km^s \quad \text{for } m = 1, 2, \dots$$

We also get similar results for other explicit constructions of  $(t, s)$  sequences.

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## 1 Introduction.

**1.1** Let  $((\beta_n^{(s)})_{n \geq 1})$  be a sequence in unit cube  $[0, 1]^s$ ,  $(\beta_{n,N}^{(s)})_{n=0}^{N-1}$  points set in  $[0, 1]^s$ ,  $J_{\mathbf{y}} = [0, y_1] \times \dots \times [0, y_s] \subseteq [0, 1]^s$ ,

$$\Delta(J_{\mathbf{y}}, (\beta_{n,N}^{(s)})_{n=0}^N) = \#\{1 \leq n \leq N \mid \beta_{n,N}^{(s)} \in J_{\mathbf{y}}\} - Ny_1 \dots y_s. \quad (1.1)$$

We define the star discrepancy of a  $(\beta_{n,N}^{(s)})_{n=0}^{N-1}$  as

$$D^*(N) = D^*((\beta_{n,N}^{(s)})_{n=0}^{N-1}) = \sup_{0 < y_1, \dots, y_s \leq 1} \left| \frac{1}{N} \Delta(J_{\mathbf{y}}, (\beta_{n,N}^{(s)})_{n=0}^N) \right|. \quad (1.2)$$

**Definition 1.** A sequence  $(\beta_n^{(s)})_{n \geq 0}$  is of **low discrepancy** (abbreviated *l.d.s.*) if  $D((\beta_n^{(s)})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$  for  $N \rightarrow \infty$ .

**Definition 2.** A sequence of point sets  $((\beta_{n,N}^{(s)})_{n=0}^{N-1})_{N=1}^{\infty}$  is of **low discrepancy** (abbreviated *l.d.p.s.*) if  $D((\beta_{n,N}^{(s)})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$ , for  $N \rightarrow \infty$ .

For examples of such a sequence, see, e.g., [BC], [DiPi], and [Ni]. In 1954, Roth proved that there exists a constant  $C_s > 0$ , such that

$$ND^*((\beta_{n,N}^{(s)})_{n=0}^{N-1}) > C_s (\ln N)^{\frac{s-1}{2}}, \quad \text{and} \quad \overline{\lim} ND^*((\beta_n^{(s)})_{n=0}^{N-1}) (\ln N)^{-s/2} > 0$$

for all  $N$ -point sets  $(\beta_{n,N}^{(s)})_{n=0}^{N-1}$  and all sequences  $(\beta_n^{(s)})_{n \geq 0}$ .

According to the well-known conjecture (see, e.g., [BC, p.283], [DiPi, p.67], [Ni, p.32]), these estimates can be improved

$$ND^*((\beta_{n,N}^{(\check{s})})_{n=0}^{N-1}) (\ln N)^{-\check{s}+1} > C'_{\check{s}} \quad \text{and} \quad \overline{\lim}_{N \rightarrow \infty} N (\ln N)^{-\check{s}} D^*((\beta_n^{(\check{s})})_{n=1}^N) > 0 \quad (1.3)$$

for all  $N$ -point sets  $(\beta_{n,N}^{(\check{s})})_{n=0}^{N-1}$  and all sequences  $(\beta_n^{(\check{s})})_{n \geq 0}$  with some  $C'_{\check{s}} > 0$ .

In 1972, W. Schmidt proved (1.3) for  $\check{s} = 1$  and  $\check{s} = 2$ . In [FaCh], (1.3) is proved for a class of  $(t, 2)$ -sequences. The review of research on this conjecture, see for example in [Bi].

A subinterval  $E$  of  $[0, 1)^s$  of the form

$$E = \prod_{i=1}^s [a_i b^{-d_i}, (a_i + 1) b^{-d_i}),$$

with  $a_i, d_i \in \mathbb{Z}$ ,  $d_i \geq 0$ ,  $0 \leq a_i < b^{d_i}$  for  $1 \leq i \leq s$  is called an *elementary interval* in base  $b \geq 2$ .

**Definition 3.** Let  $0 \leq t \leq m$  be an integer. A  $(t, m, s)$ -net in base  $b$  is a point set  $\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}$  in  $[0, 1)^s$  such that  $\#\{n \in [0, b^m - 1] | \mathbf{x}_n \in E\} = b^t$  for every elementary interval  $E$  in base  $b$  with  $\text{vol}(E) = b^{t-m}$ .

**Definition 4.** Let  $t \geq 0$  be an integer. A sequence  $\mathbf{x}_0, \mathbf{x}_1, \dots$  of points in  $[0, 1)^s$  is a  $(t, s)$ -sequence in base  $b$  if, for all integers  $k \geq 0$  and  $m \geq t$ , the point set consisting of  $\mathbf{x}_n$  with  $kb^m \leq n < (k+1)b^m$  is a  $(t, m, s)$ -net in base  $b$ .

By [Ni, p. 56,60],  $(t, m, s)$  nets and  $(t, s)$  sequences are of low discrepancy.

See reviews on  $(t, m, s)$  nets and  $(t, s)$  sequences in [DiPi] and [Ni].

For  $x = \sum_{j \geq 1} x_j p_j^{-i}$ , and  $y = \sum_{j \geq 1} y_j p_j^{-i}$  where  $x_i, y_i \in Z_b := \{0, 1, \dots, b-1\}$ , we define the  $(b$ -adic) digital shifted point  $v$  by  $v = x \oplus y := \sum_{j \geq 1} v_j p_j^{-i}$ , where  $v_i \equiv x_i + y_i \pmod{b}$  and  $v_i \in Z_b$ . For higher dimensions  $s > 1$ , let  $\mathbf{y} = (y_1, \dots, y_s) \in [0, 1]^s$ . For  $\mathbf{x} = (x_1, \dots, x_s) \in [0, 1]^s$  we define the  $(b$ -adic) digital shifted point  $\mathbf{v}$  by  $\mathbf{v} = \mathbf{x} \oplus \mathbf{y} = (x_1 \oplus y_1, \dots, x_s \oplus y_s)$ . For  $n_1, n_2 \in [0, b^m)$ , we define  $n_1 \oplus n_2 := (n_1/b^m \oplus n_2/b^m)b^m$ .

For  $x = \sum_{j \geq 1} x_j p_j^{-i}$ , where  $x_i \in Z_b$ ,  $x_i = 0$  ( $i = 1, \dots, k$ ) and  $x_{k+1} \neq 0$ , we define the absolute valuation  $\|\cdot\|_b$  of  $x$  by  $\|x\|_b = b^{-k-1}$ . Let  $\|n\|_b = b^k$  for  $n \in [b^k, b^{k+1})$ .

**Definition 5.** A point set  $(\mathbf{x}_n)_{0 \leq n < b^m}$  in  $[0, 1]^s$  is  $d$ -admissible in base  $b$  if

$$\min_{0 \leq k < n < b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > b^{-m-d} \quad \text{where} \quad \|\mathbf{x}\|_b := \prod_{i=1}^s \left\| x_j^{(i)} \right\|_b. \quad (1.4)$$

A sequence  $(\mathbf{x}_n)_{n \geq 0}$  in  $[0, 1]^s$  is  $d$ -admissible in base  $b$  if  $\inf_{n > k \geq 0} \|n \ominus k\|_b \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b \geq b^{-d}$ .

Let  $(\mathbf{x}_n)_{n \geq 0}$  be a  $d$ -admissible  $(t, s)$  sequence in base  $b$ . In [Le4], we proved for all  $m \geq 9s^2(d+t)$  that

$$1 + \max_{1 \leq N \leq b^m} ND^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < N}) \geq b^{-d} K_{d,t,s+1}^{-s} m^s \quad (1.5)$$

with some  $\mathbf{w} \in [0, 1]^s$  and  $K_{d,t,s} = 4(d+t)(s-1)^2$ .

In this paper we consider some known constructions of  $(t, s)$  sequences (e.g., Niederreiter's sequences, Xing-Niederreiter's sequences, Halton type  $(t, s)$  sequences) and we prove that they have  $d$ -admissible properties. Moreover, we prove that for these sequences the bound (1.5) is true for all  $\mathbf{w} \in [0, 1]^s$ . This result supports conjecture (1.3) (see also [Be], [LaPi], [Le2] and [Le3]).

We describe the structure of the paper. In Section 2, we fix some definitions. In Section 3, we state our results. In Section 4, we prove our outcomes.

## 2 Definitions and auxiliary results.

**4.1 Notation and terminology for algebraic function fields.** For the theory of algebraic function fields, we follow the notation and terminology in the books [St] and [Sa].

Let  $b$  be an arbitrary prime power,  $\mathbf{k} = \mathbb{F}_b$  a finite field with  $b$  elements,  $\mathbf{k}(x) = \mathbb{F}_b(x)$  the rational function field over  $\mathbb{F}_b$ , and  $\mathbf{k}[x] = \mathbb{F}_b[x]$  the polynomial ring over  $\mathbb{F}_b$ . For  $\alpha = f/g$ ,  $f, g \in \mathbf{k}[x]$ , let

$$\nu_\infty(\alpha) = \deg(g) - \deg(f) \quad (2.1)$$

be the degree valuation of  $\mathbf{k}(x)$ . We define the field of Laurent series as

$$\mathbf{k}((x)) := \left\{ \sum_{i=m}^{\infty} a_i x^i \mid m \in \mathbb{Z}, a_i \in \mathbf{k} \right\}.$$

A finite extension field  $F$  of  $\mathbf{k}(x)$  is called an algebraic function field over  $\mathbf{k}$ . Let  $\mathbf{k}$  be algebraically closed in  $F$ . We express this fact by simply saying that  $F/\mathbf{k}$  is an algebraic function field. The genus of  $F/\mathbf{k}$  is denoted by  $g$ .

A place  $\mathcal{P}$  of  $F$  is, by definition, the maximal ideal of some valuation ring of  $F$ . We denote by  $O_{\mathcal{P}}$  the valuation ring corresponding to  $\mathcal{P}$  and we denote by  $\mathbb{P}_F$  the set of places of  $F$ . For a place  $\mathcal{P}$  of  $F$ , we write  $\nu_{\mathcal{P}}$  for the normalized discrete valuation of  $F$  corresponding to  $\mathcal{P}$ , and any element  $t \in F$  with  $\nu_{\mathcal{P}}(t) = 1$  is called a local parameter (prime element) at  $\mathcal{P}$ .

The field  $F_{\mathcal{P}} := O_{\mathcal{P}}/\mathcal{P}$  is called the residue field of  $F$  with respect to  $\mathcal{P}$ . The degree of a place  $\mathcal{P}$  is defined as  $\deg(\mathcal{P}) = [F_{\mathcal{P}} : \mathbf{k}]$ . We denote by  $\text{Div}(F)$  the set of divisors of  $F/\mathbf{k}$ .

Let  $y \in F \setminus \{0\}$  and denote by  $Z(y)$ , respectively  $N(y)$ , the set of zeros, respectively poles, of  $y$ . Then we define the zero divisor of  $y$  by  $(y)_0 = \sum_{\mathcal{P} \in Z(y)} \nu_{\mathcal{P}}(y) \mathcal{P}$  and the pole divisor of  $y$  by  $(y)_{\infty} = \sum_{\mathcal{P} \in N(y)} \nu_{\mathcal{P}}(y) \mathcal{P}$ . Furthermore, the principal divisor of  $y$  is given by  $\text{div}(y) = (y)_0 - (y)_{\infty}$ .

**Theorem A (Approximation Theorem).** [St, Theorem 1.3.1] *Let  $F/\mathbf{k}$  be a function field,  $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathbb{P}_F$  pairwise distinct places of  $F/\mathbf{k}$ ,  $x_1, \dots, x_n \in F$  and  $r_1, \dots, r_n \in \mathbb{Z}$ . Then there is some  $y \in F$  such that*

$$\nu_{\mathcal{P}_i}(y - x_i) = r_i \quad \text{for } i = 1, \dots, n.$$

The completion of  $F$  with respect to  $\nu_{\mathcal{P}}$  will be denoted by  $F^{(\mathcal{P})}$ . Let  $t$  be a local parameter of  $\mathcal{P}$ . Then  $F^{(\mathcal{P})}$  is isomorphic to  $F_{\mathcal{P}}((t))$  (see [Sa, Theorem 2.5.20]), and an arbitrary element  $\alpha \in F^{(\mathcal{P})}$  can be uniquely expanded as (see [Sa, p. 293])

$$\alpha = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} S_i t^i \quad \text{where } S_i = S_i(t, \alpha) \in F_{\mathcal{P}} \subseteq F^{(\mathcal{P})}. \quad (2.2)$$

The derivative  $\frac{d\alpha}{dt}$ , or differentiation with respect to  $t$ , is defined by (see [Sa, Definition 9.3.1])

$$\frac{d\alpha}{dt} = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} i S_i t^{i-1}. \quad (2.3)$$

For an algebraic function field  $F/\mathbf{k}$ , we define its set of differentials (or Hasse differentials, H-differentials) as

$$\Delta_F = \{y \, dz \mid y \in F, \, z \text{ is a separating element for } F/\mathbf{k}\}$$

(see [St, Definition 4.1.7]).

**Proposition A.** ([St, Proposition 4.1.8] or [Sa, Theorem 9.3.13]) *Let  $z \in F$  be separating. Then every differential  $\gamma \in \Delta_F$  can be written uniquely as  $\gamma = y \, dz$  for some  $y \in F$ .*

We define the order of  $\alpha \, d\beta$  at  $\mathcal{P}$  by

$$\nu_{\mathcal{P}}(\alpha \, d\beta) := \nu_{\mathcal{P}}(\alpha \, d\beta/dt), \quad (2.4)$$

where  $t$  is any local parameter for  $\mathcal{P}$  (see [Sa, Definition 9.3.8]).

Let  $\Omega_F$  be the set of all Weil differentials of  $F/\mathbf{k}$ . There exists a  $F$ -linear isomorphism of the differential module  $\Delta_F$  onto  $\Omega_F$  (see [St, Theorem 4.3.2] or [Sa, Theorem 9.3.15]).

For  $0 \neq \omega \in \Omega_F$ , there exists a uniquely determined divisor  $\text{div}(\omega) \in \text{Div}(F)$ . Such a divisor  $\text{div}(\omega)$  is called a canonical divisor of  $F/\mathbf{k}$ . (see [St, Definition 1.5.11]). For a canonical divisor  $\dot{W}$ , we have (see [St, Corollary 1.5.16])

$$\deg(\dot{W}) = 2g - 2 \quad \text{and} \quad \ell(\dot{W}) = g. \quad (2.5)$$

Let  $\alpha \, d\beta$  be a nonzero H-differential in  $F$  and let  $\omega$  the corresponding Weil differential. Then (see [Sa, Theorem 9.3.17], [St, ref. 4.35])

$$\nu_{\mathcal{P}}(\text{div}(\omega)) = \nu_{\mathcal{P}}(\alpha \, d\beta), \quad \text{for all } \mathcal{P} \in \mathbb{P}_F. \quad (2.6)$$

Let  $\alpha \, d\beta$  be a H-differential,  $t$  a local parameter of  $\mathcal{P}$ , and

$$\alpha \, d\beta = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} S_i t^i dt \in F^{(\mathcal{P})}.$$

Then the **residue** of  $\alpha \, d\beta$  (see [Sa, Definition 9.3.10]) is defined by

$$\text{Res}_{\mathcal{P}}(\alpha \, d\beta) := \text{Tr}_{F_{\mathcal{P}}/\mathbf{k}}(S_{-1}) \in \mathbf{k}. \quad (2.7)$$

Let

$$\text{Res}_{\mathcal{P},t}(\alpha) := \text{Res}_{\mathcal{P}}(\alpha dt). \quad (2.8)$$

**Theorem B (Residue Theorem).** ([St, Corollary 4.3.3], [Sa Theorem 9.3.14]) *Let  $\alpha \, d\beta$  be any  $H$ -differential. Then  $\text{Res}_{\mathcal{P}}(\alpha \, d\beta) = 0$  for almost all places  $\mathcal{P}$ . Furthermore,*

$$\sum_{\mathcal{P} \in \mathbb{P}_F} \text{Res}_{\mathcal{P}}(\alpha \, d\beta) = 0.$$

For a divisor  $\mathcal{D}$  of  $F/\mathbf{k}$ , let  $\mathcal{L}(\mathcal{D})$  denote the Riemann-Roch space

$$\mathcal{L}(\mathcal{D}) = \mathcal{L}_F(\mathcal{D}) = \mathcal{L}_{F/\mathbf{k}}(\mathcal{D}) = \{y \in F \setminus 0 \mid \text{div}(y) + \mathcal{D} \geq 0\} \cup \{0\}.$$

Then  $\mathcal{L}(\mathcal{D})$  is a finite-dimensional vector space over  $F$ , and we denote its dimension by  $\ell(\mathcal{D})$ . By [St, Corollary 1.4.12],  $\ell(\mathcal{D}) = \{0\}$  for  $\deg(\mathcal{D}) < 0$ .

**Theorem C (Riemann-Roch Theorem).** [St, Theorem 1.5.15, and St, Theorem 1.5.17] *Let  $W$  be a canonical divisor of  $F/\mathbf{k}$ . Then for each divisor  $A \in \text{div}(F)$ ,  $\ell(A) = \deg(A) + 1 - g + \ell(W - A)$ , and*

$$\ell(A) = \deg(A) + 1 - g \quad \text{for} \quad \deg(A) \geq 2g - 1.$$

Let  $P \in \mathbb{P}_F$ ,  $e_P = \deg(P)$ , and let  $F' = FF_P$  be the **compositum** field (see [Sa, Theorem 5.4.4]). By [St, Proposition 3.6.1]  $F_P$  is the full constant field of  $F'$ .

For a place  $P \in \mathbb{P}_F$ , we define its **conorm** (with respect to  $F'/F$ ) as

$$\text{Con}_{F'/F}(P) := \sum_{P'|P} e(P'|P)P', \quad (2.9)$$

where the sum runs over all places  $P' \in \mathbb{P}_{F'}$  lying over  $P$  (see [St, Definition 3.1.8.]) and  $e(P'|P)$  is the ramification index of  $P'$  over  $P$ .

**Theorem D.** ([St, Theorem 3.6.3]) *In an algebraic constant field extension  $F' = FF_P$  of  $F/\mathbf{k}$ , the following hold:*

- (a)  $F'/F$  is unramified (i.e.,  $e(P'|P) = 1$  for all  $P \in \mathbb{P}_F$  and all  $P' \in \mathbb{P}_{F'}$  with  $P'|P$ ).
- (b)  $F'/F_P$  has the same genus as  $F/\mathbf{k}$ .
- (c) For each divisor  $A \in \text{Div}(F)$ , we have  $\deg(\text{Con}_{F'/F}(A)) = \deg(A)$ .
- (d) For each divisor  $A \in \text{Div}(F)$ ,  $\ell(\text{Con}_{F'/F}(A)) = \ell(A)$ . More precisely:  
Every basis of  $\mathcal{L}_{F/\mathbf{k}}(A)$  is also a basis of  $\mathcal{L}_{F'/F_P}(\text{Con}_{F'/F}(A))$ .

**Theorem E.** ([St, Proposition 3.1.9]) *For  $0 \neq x \in F$  let  $(x)_0^F$ ,  $(x)_\infty^F$ ,  $\text{div}(x)^F$ , resp.  $(x)_0^{F'}$ ,  $(x)_\infty^{F'}$ ,  $\text{div}(x)^{F'}$  denote the zero, pole, principal divisor of  $x$  in  $\text{Div}(F)$  resp. in  $\text{Div}(F')$ . Then*

$$\text{Con}_{F'/F}((x)_0^F) = (x)_0^{F'}, \quad \text{Con}_{F'/F}((x)_\infty^F) = (x)_\infty^{F'} \quad \text{and} \quad \text{Con}_{F'/F}(\text{div}(x)^F) = \text{div}(x)^{F'}.$$

Let  $\mathfrak{B}_1, \dots, \mathfrak{B}_\mu$  be all the places of  $F'/F_P$  lying over  $P$ . By [St, Proposition 3.1.4.], [St, Definition 3.1.5.] and Theorem D(a), we have

$$\nu_{\mathfrak{B}_i}(\alpha) = \nu_P(\alpha) \quad \text{for} \quad \alpha \in F, \quad 1 \leq i \leq \mu. \quad (2.10)$$

We will denote by  $F^{(P)}$  resp.  $F'^{(\mathfrak{B}_i)}$  ( $1 \leq i \leq \mu$ ) the completion of  $F$  resp.  $F'$  with respect to the valuation  $\nu_P$  resp.  $\nu_{\mathfrak{B}_i}$ . Applying [Sa, p.132, 133], we obtain

$$F \subseteq F^{(P)} \subseteq F'^{(\mathfrak{B}_i)} \quad \text{and} \quad F \subseteq F' \subseteq F'^{(\mathfrak{B}_i)}, \quad 1 \leq i \leq \mu.$$

Let  $t$  be a local parameter of  $\mathcal{P}$ , and let  $\alpha \in F^{(P)}$ . By (2.10), we have  $\nu_{\mathfrak{B}_i}(t) = 1$ . Consider the local expansion (2.2). Using (2.10), we get  $\nu_{\mathfrak{B}_i}(\alpha) = \nu_P(\alpha)$ . Hence

$$\nu_{\mathfrak{B}_i}(\alpha) = \nu_P(\alpha) \quad \text{for} \quad \alpha \in F' \cap F^{(P)} \quad 1 \leq i \leq \mu. \quad (2.11)$$

**Theorem F.** ([LiNi, Theorem 2.24]) *Let  $M$  be a finite extension of the finite field  $L$ , both considered as vector spaces over  $L$ . Then the linear transformations from  $M$  into  $L$  are exactly the mappings  $K_\beta$ ,  $\beta \in F$  where  $K_\beta = \text{Tr}_{M/L}(\beta\alpha)$  for all  $\alpha \in F$ . Furthermore, we have  $K_\beta \neq K_\gamma$  whenever  $\beta$  and  $\gamma$  are distinct elements of  $L$ .*

**Theorem G.** ([St, Proposition 3.3.3] or [LiNi, Definition 2.30, and p.58]) *Let  $L$  be a finite field and  $M$  a finite extension of  $L$ . Consider a basis  $\{\alpha_1, \dots, \alpha_m\}$  of  $M/L$ . Then there are uniquely determined elements  $\beta_1, \dots, \beta_m$  of  $M$ , such that*

$$\text{Tr}_{M/L}(\alpha_i \beta_j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (2.12)$$

*The set  $\beta_1, \dots, \beta_m$  is a basis of  $M/L$  as well; it is called the dual basis of  $\{\alpha_1, \dots, \alpha_m\}$  (with respect to the trace).*

#### 4.2 Digital sequences and $(\mathbf{T}, s)$ sequences ([DiPi, Section 4]).

**Definition 6.** ([DiPi, Definition 4.30]) *For a given dimension  $s \geq 1$ , an integer base  $b \geq 2$ , and a function  $\mathbf{T} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  with  $\mathbf{T}(m) \leq m$  for all  $m \in \mathbb{N}_0$ , a sequence  $(\mathbf{x}_0, \mathbf{x}_1, \dots)$  of points in  $[0, 1)^s$  is called a  $(\mathbf{T}, s)$ -sequence in base  $b$  if for all integers  $m \geq 0$  and  $k \geq 0$ , the point set consisting of the points  $x_{kb^m}, \dots, x_{kb^m+b^m-1}$  forms a  $(\mathbf{T}(m), m, s)$ -net in base  $b$ .*

**Lemma A.** ([DiPi, Lemma 4.38]) *Let  $(\mathbf{x}_0, \mathbf{x}_1, \dots)$  be a  $(\mathbf{T}, s)$ -sequence in base  $b$ . Then, for every  $m$ , the point set  $\{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{b^m-1}\}$  with  $\mathbf{y}_k := (\mathbf{x}_k, k/b^m)$ ,  $0 \leq k < b^m$ , is an  $(r(m), m, s+1)$ -net in base  $b$  with  $r(m) := \max\{\mathbf{T}(0), \dots, \mathbf{T}(m)\}$ .*

Repeating the proof of this lemma, we obtain

**Lemma 1.** *Let  $(\mathbf{x}_n)_{n \geq 0}$  be a sequence in  $[0, 1)^s$ ,  $m_n \in \mathbb{N}$ ,  $m_i > m_j$  for  $i > j$ , and let  $(\mathbf{x}_n, n/b^{m_k})_{0 \leq n < b^{m_k}}$  be a  $(t, m_k, s+1)$  net in base  $b$  for all  $k \geq 1$ . Then  $(\mathbf{x}_n)_{n \geq 0}$  is a  $(t, s)$  sequence in base  $b$ .*

**Lemma B.** ([Ni, Lemma 3.7]) *Let  $(\mathbf{x}_n)_{n \geq 0}$  be a sequence in  $[0, 1)^s$ . For  $N \geq 1$ , let  $H$  be the point set consisting of  $(\mathbf{x}_n, n/N) \in [0, 1)^{s+1}$  for  $n = 0, \dots, N-1$ . Then*

$$1 + \max_{1 \leq M \leq N} MD^*((\mathbf{x}_n)_{n=0}^{M-1}) \geq ND^*((\mathbf{x}_n, n/N)_{n=0}^{N-1}).$$

**Definition 7.** ([DiNi, Definition 1]) *Let  $m, s \geq 1$  be integers. Let  $C^{(1,m)}, \dots, C^{(s,m)}$  be  $m \times m$  matrices over  $\mathbb{F}_b$ . Now we construct  $b^m$  points in  $[0, 1)^s$ . For  $n = 0, 1, \dots, b^m - 1$ , let  $n = \sum_{j=0}^{m-1} a_j(n)b^j$  be the  $b$ -adic expansion of  $n$ . Choose a*



bijection  $\phi : Z_b := \{0, 1, \dots, b-1\} \mapsto \mathbb{F}_b$  with  $\phi(0) = \bar{0}$ , the neutral element of addition in  $\mathbb{F}_b$ . Let  $|\phi(a)| := |a|$  for  $a \in Z_b$ . We identify  $n$  with the row vector

$$\mathbf{n} = (\bar{a}_0(n), \dots, \bar{a}_{m-1}(n)) \in \mathbb{F}_b^m \quad \text{with} \quad \bar{a}_i(n) = \phi(a_i(n)), \quad 0 \leq i \leq m-1. \quad (2.13)$$

We map the vectors

$$y_n^{(i)} = (y_{n,1}^{(i)}, \dots, y_{n,m}^{(i)}) := \mathbf{n} C^{(i,m)\top} \in \mathbb{F}_b^m \quad (2.14)$$

to the real numbers

$$x_n^{(i)} = \sum_{j=1}^m \phi^{-1}(y_{n,j}^{(i)})/b^j \quad (2.15)$$

to obtain the point

$$\mathbf{x}_n := (x_n^{(1)}, \dots, x_n^{(s)}) \in [0, 1]^s. \quad (2.16)$$

The point set  $\{\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}\}$  is called a **digital net** (over  $\mathbb{F}_b$ ) (with generating matrices  $(C^{(1,m)}, \dots, C^{(s,m)})$ ).

For  $m = \infty$ , we obtain a sequence  $\mathbf{x}_0, \mathbf{x}_1, \dots$  of points in  $[0, 1]^s$  which is called a **digital sequence** (over  $\mathbb{F}_b$ ) (with generating matrices  $(C^{(1,\infty)}, \dots, C^{(s,\infty)})$ ).

We abbreviate  $C^{(i,m)}$  as  $C^{(i)}$  for  $m \in \mathbb{N}$  and for  $m = \infty$ .

### 4.3 Duality theory ( see [DiPi, Section 7], [DiNi], [NiPi], [Skr]).

Let  $\mathcal{N}$  be an arbitrary  $\mathbb{F}_b$ -linear subspace of  $F_b^{sm}$ . Let  $H$  be a matrix over  $\mathbb{F}_b$  consisting of  $sm$  columns such that the row-space of  $H$  is equal to  $\mathcal{N}$ . Then we define the **dual space**  $\mathcal{N}^\perp \subseteq F_b^{sm}$  of  $\mathcal{N}$  to be the null space of  $H$  (see [DiPi, p. 244]). In other words,  $\mathcal{N}^\perp$  is the orthogonal complement of  $\mathcal{N}$  relative to the standard inner product in  $F_b^{sm}$ ,

$$\mathcal{N}^\perp = \{A \in F_b^{sm} \mid B \cdot A = 0 \quad \text{for all } B \in \mathcal{N}\}. \quad (2.17)$$

For any vector  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}_b^m$ , let

$$v(\mathbf{a}) = 0 \quad \text{if } \mathbf{a} = \mathbf{0} \quad \text{and} \quad v_m(\mathbf{a}) = \max\{j : a_j \neq 0\} \quad \text{if } \mathbf{a} \neq \mathbf{0}.$$

Then we extend this definition to  $\mathbb{F}_b^{ms}$  by writing a vector  $\mathbf{A} \in \mathbb{F}_b^{ms}$  as the concatenation of  $s$  vectors of length  $m$ , i.e.  $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbb{F}_b^{ms}$  with  $\mathbf{a}_i \in \mathbb{F}_b^m$  for  $1 \leq i \leq s$  and putting

$$V_m(\mathbf{A}) = \sum_{1 \leq i \leq s} v_m(\mathbf{a}_i). \quad (2.18)$$

**Definition 8.** For any nonzero  $\mathbb{F}_b^m$ -linear subspace  $\mathcal{N}$  of  $\mathbb{F}_b^{ms}$ , the minimum distance of  $\mathcal{N}$  is defined by

$$\delta_m(\mathcal{N}) = \min\{V_m(\mathbf{A}) \mid \mathbf{A} \in \mathcal{N} \setminus \{\mathbf{0}\}\}.$$

Let  $C^{(1)}, \dots, C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$  be generating matrices of a digital sequence  $\mathbf{x}_n(C)_{n \geq 0}$  over  $\mathbb{F}_b$ . For any  $m \in \mathbb{N}$ , we denote the  $m \times m$  left-upper sub-matrix of  $C^{(i)}$  by  $[C^{(i)}]_m$ . The matrices  $[C^{(1)}]_m, \dots, [C^{(s)}]_m$  are then the generating matrices of a digital net. We define the **overall generating matrix** of this digital net by

$$[C]_m = ([C^{(1)}]_m^\top | [C^{(2)}]_m^\top | \dots | [C^{(s)}]_m^\top) \in F_b^{m \times sm} \quad (2.19)$$

for any  $m \in \mathbb{N}$ .

Let  $\mathcal{C}_m$  denote the row space of the matrix  $[C]_m$  i.e.,

$$\mathcal{C}_m = \left\{ \left( \sum_{r=0}^{m-1} c_{j,r}^{(i)} \bar{a}_r(n) \right)_{0 \leq j \leq m-1, 1 \leq i \leq s} \mid 0 \leq n < b^m \right\}. \quad (2.20)$$

The dual space is then given by

$$\mathcal{C}_m^\perp = \{A \in F_b^{sm} \mid B \cdot A = \mathbf{0} \text{ for all } B \in \mathcal{C}_m\}. \quad (2.21)$$

**Proposition B.** ([DiPi, Proposition 7.22]) *For  $s \in \mathbb{N}$ ,  $s \geq 2$ , the matrices  $C^{(1)}, \dots, C^{(s)}$  generate a digital  $(\mathbf{T}, s)$ -sequence if and only if for all  $m \in \mathbb{N}$  we have*

$$\mathbf{T}(m) \geq m - \delta_m(\mathcal{C}_m^\perp) + 1, \quad \text{for all } m \in \mathbb{N}.$$

We define a weight function on  $\mathbb{F}_b^{ms}$  dual to the weight function  $V_m$  (2.18). For any vector  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}_b^m$ , let

$$v^\perp(\mathbf{a}) = m + 1 \text{ if } \mathbf{a} = \mathbf{0} \text{ and } v_m^\perp(\mathbf{a}) = \min\{j : a_j \neq 0\} \text{ if } \mathbf{a} \neq \mathbf{0}. \quad (2.22)$$

Then we extend this definition to  $\mathbb{F}_b^{ms}$  by writing a vector  $\mathbf{A} \in \mathbb{F}_b^{ms}$  as the concatenation of  $s$  vectors of length  $m$ , i.e.  $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbb{F}_b^{ms}$  with  $\mathbf{a}_i \in \mathbb{F}_b^m$  for  $1 \leq i \leq s$  and putting

$$V_m^\perp(\mathbf{A}) = \sum_{1 \leq i \leq s} v_m^\perp(\mathbf{a}_i). \quad (2.23)$$

**Definition 9.** For any nonzero  $\mathbb{F}_b^m$ -linear subspace  $\mathcal{N}$  of  $\mathbb{F}_b^{ms}$ , the maximum distance of  $\mathcal{N}$  is defined by

$$\delta_m^\perp(\mathcal{N}) = \max\{V_m^\perp(\mathbf{A}) \mid \mathbf{A} \in \mathcal{N} \setminus \{\mathbf{0}\}\}. \quad (2.24)$$

#### 4.4 Auxiliary results.

**Lemma C.** ([Le4, Lemma 1]) *Let  $\dot{s} \geq 2$ ,  $d \geq 1$ ,  $(\mathbf{x}_n)_{0 \leq n < b^{\tilde{m}}}$  be a  $d$ -admissible  $(t, \tilde{m}, \dot{s})$  net in base  $b$ ,  $d_0 = d + t$ ,  $\hat{e} \in \mathbb{N}$ ,  $0 < \epsilon \leq (2d_0\hat{e}(\dot{s} - 1))^{-1}$ ,  $\dot{m} = [\tilde{m}\epsilon]$ ,  $\ddot{m}_i = 0$ ,  $\dot{m}_i = d_0\hat{e}\dot{m}$  ( $1 \leq i \leq \dot{s} - 1$ ),  $\ddot{m}_{\dot{s}} = \tilde{m} - (\dot{s} - 1)\dot{m}_1 - t \geq 1$ ,  $\dot{m}_{\dot{s}} = \ddot{m}_{\dot{s}} + \dot{m}_1$ ,  $B_i \subset \{0, \dots, \dot{m} - 1\}$  ( $1 \leq i \leq \dot{s}$ ),  $\mathbf{w} \in E_{\tilde{m}}^{\dot{s}}$  and let  $\gamma^{(i)} = \gamma_1^{(i)}/b + \dots + \gamma_{\tilde{m}_i}^{(i)}/b^{\tilde{m}_i}$ ,*

$$\gamma_{\tilde{m}_i + d_0(\hat{j}_i\hat{e} + \check{j}_i) + \check{j}_i}^{(i)} = 0 \text{ for } 1 \leq \check{j}_i < d_0, \quad \gamma_{\tilde{m}_i + d_0(\hat{j}_i\hat{e} + \check{j}_i) + \check{j}_i}^{(i)} = 1 \text{ for } \check{j}_i = d_0 \quad (2.25)$$

*and  $\hat{j}_i \in \{0, \dots, \dot{m} - 1\} \setminus B_i$ ,  $0 \leq \check{j}_i < \hat{e}$ ,  $1 \leq i \leq \dot{s}$ ,  $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(\dot{s})})$ ,  $B = \#B_1 + \dots + \#B_{\dot{s}}$  and  $\tilde{m} \geq 4\epsilon^{-1}(\dot{s} - 1)(1 + \dot{s}B) + 2t$ . Let there exists  $n_0 \in [0, b^{\tilde{m}})$  such that  $[(\mathbf{x}_{n_0} \oplus \mathbf{w})^{(i)}]_{\dot{m}_i} = \gamma^{(i)}$ ,  $1 \leq i \leq \dot{s}$ . Then*

$$\Delta((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}, J_{\boldsymbol{\gamma}}) \leq -b^{-d}(\hat{e}\epsilon(2(\dot{s} - 1))^{-1})^{\dot{s}-1} \tilde{m}^{\dot{s}-1} + b^{t+s} d_0 \hat{e} B \tilde{m}^{\dot{s}-2}. \quad (2.26)$$

**Corollary 1.** *With notations as above. Let  $\dot{s} \geq 3$ ,  $\tilde{r} \geq 0$ ,  $\tilde{m} = m - \tilde{r}$ ,  $(\mathbf{x}_n)_{0 \leq n < b^{\tilde{m}}}$  be a  $d$ -admissible  $(t, \tilde{m}, \dot{s})$  net in base  $b$ ,  $d_0 = d + t$ ,  $\hat{e} \in \mathbb{N}$ ,  $\epsilon = \eta(2d_0\hat{e}(\dot{s} - 1))^{-1}$ ,  $0 < \eta \leq 1$ ,  $\dot{m} = [\tilde{m}\epsilon]$ ,  $\ddot{m}_i = 0$ ,  $\dot{m}_i = d_0\hat{e}\dot{m}$ ,  $\ddot{m}_{\dot{s}} = \tilde{m} - (\dot{s} - 1)\dot{m}_1 - t \geq 1$ ,  $\dot{m}_{\dot{s}} = \ddot{m}_{\dot{s}} + \dot{m}_1$ ,  $B_i \subset \{0, \dots, \dot{m} - 1\}$ ,  $\bar{B}_i = \{0, \dots, \dot{m} - 1\} \setminus B_i$ ,  $1 \leq i \leq \dot{s}$ ,  $B = \#B_1 + \dots + \#B_{\dot{s}}$ . Suppose that*

$$\{(x_{n, \tilde{m}_i + d_0\hat{e}\hat{j}_i + \check{j}_i}^{(i)} \mid \hat{j}_i \in \bar{B}_i, \check{j}_i \in [1, d_0\hat{e}], i \in [1, \dot{s}]) \mid n \in [0, b^{\tilde{m}})\} = Z_b^\mu, \quad (2.27)$$

*with  $m \geq 2t + 8(d + t)\hat{e}(\dot{s} - 1)^2\eta^{-1} + 2^{2\dot{s}}b^{d+\dot{s}+t}(d + t)^{\dot{s}}\hat{e}(\dot{s} - 1)^{2(\dot{s}-1)}\eta^{-\dot{s}+1}B + 4(\dot{s} - 1)\tilde{r}$  and  $\mu = d_0\hat{e}(\dot{s}\dot{m} - B)$ . Then there exists  $n_0 \in [0, b^{\tilde{m}})$  such that  $[(\mathbf{x}_{n_0} \oplus \mathbf{w})^{(i)}]_{\dot{m}_i} = \gamma^{(i)}$ ,  $1 \leq i \leq \dot{s}$ , and for each  $\mathbf{w} \in E_{\tilde{m}}^{\dot{s}}$ , we have*

$$b^{\tilde{m}} D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}) \geq \left| \Delta((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}, J_{\boldsymbol{\gamma}}) \right| \geq 2^{-2} b^{-d} K_{d,t,\dot{s}}^{-\dot{s}+1} \eta^{\dot{s}-1} m^{\dot{s}-1}$$

with  $K_{d,t,\dot{s}} = 4(d + t)(\dot{s} - 1)^2$ .

**Proof.** Let  $\gamma(n, \mathbf{w}) = \gamma = (\gamma^{(1)}, \dots, \gamma^{(\dot{s})})$  with  $\gamma^{(i)} := [(\mathbf{x}_n \oplus \mathbf{w})^{(i)}]_{\dot{m}_i}$ ,  $i \in [1, \dot{s}]$ . Using (2.27), we get that there exists  $n_0 \in [0, b^{\tilde{m}})$  such that  $\gamma(n_0, \mathbf{w})$  satisfy (2.25). Hence (2.26) is true. Taking into account (1.2) and that  $\mathbf{w} \in E_{\tilde{m}}^{\dot{s}}$  is arbitrary, we get the assertion in Corollary 1.  $\square$

Let  $\phi : Z_b \mapsto \mathbb{F}_b$  be a bijection with  $\phi(0) = \bar{0}$ , and let  $x_{n,j}^{(i)} = \phi^{-1}(y_{n,j}^{(i)})$  for  $1 \leq i \leq \dot{s}$ ,  $j \geq 1$  and  $n \geq 0$ . We obtain from Corollary 1 :

**Corollary 2.** Let  $\dot{s} \geq 3$ ,  $\tilde{r} \geq 0$ ,  $\tilde{m} = m - \tilde{r}$ ,  $(\mathbf{x}_n)_{0 \leq n < b^{\tilde{m}}}$  be a  $d$ -admissible  $(t, \tilde{m}, \dot{s})$  net in base  $b$ ,  $d_0 = d + t$ ,  $\hat{e} \in \mathbb{N}$ ,  $\epsilon = \eta(2d_0\hat{e}(\dot{s} - 1))^{-1}$ ,  $0 < \eta \leq 1$ ,  $\dot{m} = [\tilde{m}\epsilon]$ ,  $\ddot{m}_i = 0$ ,  $\dot{m}_i = d_0\hat{e}\dot{m}$ ,  $\ddot{m}_{\dot{s}} = \tilde{m} - (\dot{s} - 1)\dot{m}_1 - t \geq 1$ ,  $\dot{m}_{\dot{s}} = \ddot{m}_{\dot{s}} + \dot{m}_1$ ,  $B_i \subset \{0, \dots, \dot{m} - 1\}$ ,  $\bar{B}_i = \{0, \dots, \dot{m} - 1\} \setminus B_i$ ,  $1 \leq i \leq \dot{s}$ ,  $B = \#B_1 + \dots + \#B_{\dot{s}}$ . Suppose that

$$\{(y_{n, \dot{m}_i + d_0\hat{e}\check{j}_i + \check{j}_i}^{(i)} \mid \hat{j}_i \in \bar{B}_i, \check{j}_i \in [1, d_0\hat{e}], i \in [1, \dot{s}]) \mid n \in [0, b^m)\} = \mathbb{F}_b^\mu,$$

with  $m \geq 2t + 8(d + t)\hat{e}(\dot{s} - 1)^2\eta^{-1} + 2^{2\dot{s}}b^{d+\dot{s}+t}(d + t)^{\dot{s}}\hat{e}(\dot{s} - 1)^{2(\dot{s}-1)}\eta^{-\dot{s}+1}B + 4(\dot{s} - 1)\tilde{r}$  and  $\mu = d_0\hat{e}(\dot{s}\dot{m} - B)$ . Then there exists  $n_0 \in [0, b^{\tilde{m}})$  such that  $[(\mathbf{x}_{n_0} \oplus \mathbf{w})^{(i)}]_{\dot{m}_i} = \gamma^{(i)}$ ,  $1 \leq i \leq \dot{s}$ , and for each  $\mathbf{w} \in E_{\tilde{m}}^{\dot{s}}$ , we have

$$b^{\tilde{m}}D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}) \geq \left| \Delta((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^{\tilde{m}}}, J_\gamma) \right| \geq 2^{-2}b^{-d}K_{d,t,\dot{s}}^{-\dot{s}+1}\eta^{\dot{s}-1}m^{\dot{s}-1}.$$

With notations as above, we consider the case of  $(t, s)$  sequence in base  $b$ :

**Corollary 3.** Let  $s \geq 2$ ,  $d \geq 1$ ,  $(\mathbf{x}_n)_{n \geq 0}$  be a  $d$ -admissible  $(t, s)$  sequence in base  $b$ ,  $d_0 = d + t$ ,  $\hat{e} \in \mathbb{N}$ ,  $\epsilon = \eta(2d_0\hat{e}s)^{-1}$ ,  $0 < \eta \leq 1$ ,  $\dot{m} = [m\epsilon]$ ,  $\ddot{m}_i = 0$ ,  $1 \leq i \leq s$ ,  $\ddot{m}_{s+1} = t - 1 + (s - 1)d_0\hat{e}\dot{m}$ ,  $B'_i \subset \{0, \dots, \dot{m} - 1\}$ ,  $\bar{B}'_i = \{0, \dots, \dot{m} - 1\} \setminus B'_i$ ,  $1 \leq i \leq s + 1$ ,  $B = \#B'_1 + \dots + \#B'_{s+1}$ . Suppose that

$$\{(y_{n, \ddot{m}_i + d_0\hat{e}\check{j}_i + \check{j}_i}^{(i)} \mid \hat{j}_i \in \bar{B}'_i, \check{j}_i \in [1, d_0\hat{e}], i \in [1, s], \\ \bar{a}_{\ddot{m}_{s+1} + d_0\hat{e}\check{j}_{s+1} + \check{j}_{s+1}}(n), \check{j}_{s+1} \in \bar{B}'_{s+1}, \check{j}_{s+1} \in [1, d_0\hat{e}], ) \mid n \in [0, b^m)\} = \mathbb{F}_b^\mu.$$

with  $\mu = d_0\hat{e}((s + 1)\dot{m} - B)$ , and  $m \geq 2t + 8(d + t)\hat{e}s^2\eta^{-1} + 2^{2s+2}b^{d+s+t+1}(d + t)^{s+1}\hat{e}s^{2s}\eta^{-s}B$ . Then

$$1 + \min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E_m^s} \max_{1 \leq N \leq b^m} ND^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w})_{0 \leq n < N}) \geq 2^{-2}b^{-d}K_{d,t,s+1}^{-s}\eta^s m^s.$$

**Proof.** Using Lemma B, we have

$$\begin{aligned} 1 + \sup_{1 \leq N \leq b^m} ND^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w})_{0 \leq n < N}) &\geq b^m D^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w}, n/b^m)_{0 \leq n < b^m}) \\ &= b^m D^*((\mathbf{x}_n \oplus \mathbf{w}, (n \ominus Q)/b^m)_{0 \leq n < b^m}). \end{aligned}$$

By (1.4) and [DiPi, Lemma 4.38], we have that  $((\mathbf{x}_n, n/b^m)_{0 \leq n < b^m})$  is a  $d$ -admissible  $(t, m, s+1)$ -net in base  $b$ . We apply Corollary 2 with  $\dot{s} = s+1$ ,  $\tilde{r} = 0$ ,  $B'_i = B_i$ ,  $1 \leq i < \dot{s}$ ,  $B'_s = \{\dot{m} - j - 1 | j \in B_s\}$ ,  $\hat{j}_{s+1} = \dot{m} - \check{j}_{s+1} - 1$ ,  $\check{j}_{s+1} = d_0 \hat{e} - \check{j}_{s+1} + 1$ , and  $x_n^{(s+1)} = n/b^m$ . Taking into account that  $y_{n, m-j}^{(s+1)} = \bar{a}_j(n)$  ( $0 \leq j < m$ ), we get  $y_{n, m-\check{m}_{s+1}-d_0 \hat{e} \dot{m}-1+d_0 \hat{e} \check{j}_{s+1}+\check{j}_{s+1}}^{(s+1)} = \bar{a}_{\check{m}_{s+1}+d_0 \hat{e} \check{j}_{s+1}+\check{j}_{s+1}}(n)$ , and Corollary 3 follows.  $\square$

**Lemma 2.** Let  $\dot{s} \geq 2$ ,  $d_0 \geq 1$ ,  $\hat{e} \geq 1$ ,  $\dot{m} \geq 1$ ,  $\dot{m}_1 = d_0 \hat{e} \dot{m}$ ,  $\ddot{m}_i \in [0, m - \dot{m}_1]$  ( $1 \leq i \leq \dot{s}$ ),  $m \geq \dot{s} \dot{m}_1$ ,  $\dot{m} \geq r$ , and let

$$\Phi := \{(y_{n, \check{m}_1+1}^{(1)}, \dots, y_{n, \check{m}_1+\dot{m}_1}^{(1)}, \dots, y_{n, \check{m}_s+1}^{(\dot{s})}, \dots, y_{n, \check{m}_s+\dot{m}_1}^{(\dot{s})}) | n \in [0, b^m)\} \subseteq \mathbb{F}_b^{\dot{s} \dot{m}_1}. \quad (2.28)$$

Suppose that  $\Phi$  is a  $\mathbb{F}_b$  linear subspace of  $\mathbb{F}_b^{\dot{s} \dot{m}_1}$  and  $\dim_{\mathbb{F}_b}(\Phi) = \dot{s} \dot{m}_1 - r$ . Then there exists  $B_i \in \{0, \dots, \dot{m} - 1\}$ ,  $1 \leq i \leq \dot{s}$ , with  $B = \#B_1 + \dots + \#B_s \leq r$  and

$$\Psi = \mathbb{F}_b^{d_0 \hat{e} (\dot{s} \dot{m} - B)}, \quad (2.29)$$

where

$$\Psi = \{(y_{n, \check{m}_i+d_0 \hat{e} (\dot{j}_i-1)+\check{j}_i}^{(i)} | \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, d_0 \hat{e}], i \in [1, \dot{s}]) | n \in [0, b^m)\} \quad (2.30)$$

with  $\bar{B}_i = \{0, \dots, \dot{m} - 1\} \setminus B_i$ .

**Proof.** Let  $\hat{r} = \dot{s} \dot{m}_1 - r$ , and let  $\mathbf{f}_1, \dots, \mathbf{f}_{\hat{r}}$  be a basis of  $\Phi$  with

$$\mathbf{f}_\mu = (f_{\mu, \check{m}_1+1}^{(1)}, \dots, f_{\mu, \check{m}_1+\dot{m}_1}^{(1)}, \dots, f_{\mu, \check{m}_s+1}^{(\dot{s})}, \dots, f_{\mu, \check{m}_s+\dot{m}_1}^{(\dot{s})}), \quad 1 \leq \mu \leq \hat{r}.$$

Let

$$v(\mathbf{f}_\mu) = \max \{\ddot{m}_i + (i-1)\dot{m}_1 + j | f_{\mu, \check{m}_i+j}^{(i)} \neq 0, j \in [1, \dot{m}_1], i \in [1, \dot{s}]\} \text{ for } \mu \in [1, \hat{r}].$$

Without loss of generality, assume now that  $v(\mathbf{f}_i) \leq v(\mathbf{f}_j)$  for  $1 \leq i < j \leq \hat{r}$ . Let  $v(\mathbf{f}_j) = \ddot{m}_{l_1} + (l_1-1)\dot{m}_1 + l_2$ , and let  $\dot{\mathbf{f}}_k = \mathbf{f}_k - \mathbf{f}_j f_{k, \check{m}_{l_1}+l_2}^{(l_1)} / f_{j, \check{m}_{l_1}+l_2}^{(l_1)}$  for  $1 \leq k \leq j-1$ . We have  $v(\dot{\mathbf{f}}_k) < v(\mathbf{f}_j)$  for all  $1 \leq k \leq j-1$ .

By repeating this procedure for  $j = \hat{r}, \hat{r} - 1, \dots, 2$ , we obtain a basis  $\hat{\mathbf{f}}_1, \dots, \hat{\mathbf{f}}_{\hat{r}}$  of  $\Phi$  with  $v(\hat{\mathbf{f}}_i) < v(\hat{\mathbf{f}}_j)$  for  $1 \leq i < j \leq \hat{r}$ . Let

$$A_i = \{\ddot{m}_i + j \mid v(\hat{\mathbf{f}}_\mu) = (i-1)\dot{m}_1 + \ddot{m}_i + j, 1 \leq j \leq \dot{m}_1, 1 \leq \mu \leq \hat{r}\}, i \in [1, \dot{s}].$$

Taking into account that  $\hat{\mathbf{f}}_1, \dots, \hat{\mathbf{f}}_{\hat{r}}$  is a basis of  $\Phi$ , we get from (2.28)

$$\{(y_{n,j}^{(i)} \mid j \in A_i, i \in [1, \dot{s}]) \mid n \in [0, b^m]\} = \mathbb{F}_b^{\dot{s}\dot{m}_1 - r}. \quad (2.31)$$

Now let

$$\bar{B}_i := \{\dot{j}_i \in [0, \dot{m}_1] \mid \exists \ddot{j}_i \in [1, d_0\hat{e}], \text{ with } \ddot{m}_i + \dot{j}_i d_0\hat{e} + \ddot{j}_i \in A_i\}, i \in [1, \dot{s}].$$

It is easy to see that  $B = \#B_1 + \dots + \#B_{\dot{s}} \leq r$ , where  $\bar{B}_i = \{0, \dots, \dot{m}_1 - 1\} \setminus B_i$ .

Bearing in mind (2.30), we obtain (2.29) from (2.31). Hence Lemma 2 is proved.  $\square$

### 3 Statements of results.

If  $s = 2$  for the case of nets, or  $s = 1$  for the case of sequences, then (1.5) follows from the W. Schmidt estimate (1.3) (see [Ni, p.24]). In this paper we take  $s \geq 2$  for the case of sequences, and  $s \geq 3$  for the case of nets.

**2.1 Generalized Niederreiter sequence.** In this subsection, we introduce a generalization of the Niederreiter sequence due to Tezuka (see [Te2, Section 6.1.2], [DiPi, Section 8.1.2]). By [Te2, p.165], the Sobol's sequence [DiPi, Section 8.1.2], the Faure's sequence [DiPi, Section 8.1.2]) and the original Niederreiter sequence [DiPi, Section 8.1.2]) are particular cases of a generalized Niederreiter sequence.

Let  $b$  be a prime power and let  $p_1, \dots, p_s \in F_b[x]$  be pairwise coprime polynomials over  $\mathbb{F}_b$ . Let  $e_i = \deg(p_i) \geq 1$  for  $1 \leq i \leq s$ . For each  $j \geq 1$  and  $1 \leq i \leq s$ , the set of polynomials  $\{y_{i,j,k}(x) : 0 \leq k < e_i\}$  needs to be linearly independent (mod  $p_i(x)$ ) over  $\mathbb{F}_b$ . For integers  $1 \leq i \leq s$ ,  $j \geq 1$  and  $0 \leq k < e_i$ , consider the expansions

$$\frac{y_{i,j,k}(x)}{p_i(x)^j} = \sum_{r \geq 0} a^{(i)}(j, k, r) x^{-r-1} \quad (3.1)$$

over the field of formal Laurent series  $F_b((x^{-1}))$ . Then we define the matrix  $C^{(i)} = (c_{j,r}^{(i)})_{j \geq 1, r \geq 0}$  by

$$c_{j,r}^{(i)} = a^{(i)}(Q+1, k, r) \in \mathbb{F}_b \quad \text{for} \quad 1 \leq i \leq s, j \geq 1, r \geq 0,$$

where  $j-1 = Qe_i + k$  with integers  $Q = Q(i, j)$  and  $k = k(i, j)$  satisfying  $0 \leq k < e_i$ .

A digital sequence  $(\mathbf{x}_n)_{n \geq 0}$  over  $\mathbb{F}_b$  generated by the matrices  $C^{(1)}, \dots, C^{(s)}$  is called a **generalized Niederreiter sequence** (see [DiPi, p.266]).

**Theorem H.** (see [DiPi, p.266]) *The generalized Niederreiter sequence with generating matrices, defined as above, is a digital  $(t, s)$ -sequence over  $\mathbb{F}_b$  with  $t = e_0 - s$  and  $e_0 = e_1 + \dots + e_s$ .*

**Theorem 1.** *With the notations as above,  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible with  $d = e_0$ . (a) For  $s \geq 2$ ,  $e = e_1 e_2 \dots e_s$ ,  $\eta_1 = s/(s+1)$ ,  $m \geq 9(d+t)es(s+1)$  and  $K_{d,t,s} = 4(d+t)(s-1)^2$ , we have*

$$1 + \min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E_m^s} \max_{1 \leq N \leq b^m} ND^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < N}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} \eta_1^s m^s.$$

(b) Let  $s \geq 3$ ,  $\eta_2 \in (0, 1)$  and  $m \geq 8(d+t)e(s-1)^2 \eta_2^{-1} + 2(1+t) \eta_2^{-1} (1-\eta_2)^{-1}$ . Suppose that  $\min_{m/2-t \leq j e_{i_0} \leq m, 0 \leq k < e_{i_0}} (1 - \deg(y_{i_0,j,k}(x)) j^{-1} e_{i_0}^{-1}) \geq \eta_2$  for some  $i_0 \in [1, s]$ . Then

$$\min_{\mathbf{w} \in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^m}) \geq 2^{-2} b^{-d} K_{d,t,s}^{-s+1} \eta_2^{s-1} m^{s-1}.$$

**2.2 Xing-Niederreiter sequence** (see [DiPi, Section 8.4]). Let  $F/\mathbb{F}_b$  be an algebraic function field with full constant field  $\mathbb{F}_b$  and genus  $g = g(F/\mathbb{F}_b)$ . Assume that  $F/\mathbb{F}_b$  has at least one rational place  $P_\infty$ , and let  $G$  be a positive divisor of  $F/\mathbb{F}_b$  with  $\deg(G) = 2g$  and  $P_\infty \notin \text{supp}(G)$ . Let  $P_1, \dots, P_s$  be  $s$  distinct places of  $F/\mathbb{F}_b$  with  $P_i \neq P_\infty$  for  $1 \leq i \leq s$ . Put  $e_i = \deg(P_i)$  for  $1 \leq i \leq s$ .

By [DiPi, p.279], we have that there exists a basis  $w_0, w_1, \dots, w_g$  of  $\mathcal{L}(G)$  over  $\mathbb{F}_b$  such that

$$\nu_{P_\infty}(w_u) = n_u \quad \text{for } 0 \leq u \leq g,$$

where  $0 = n_0 < n_1 < \dots < n_g \leq 2g$ . For each  $1 \leq i \leq s$ , we consider the chain

$$\mathcal{L}(G) \subset \mathcal{L}(G + P_i) \subset \mathcal{L}(G + 2P_i) \subset \dots$$

of vector spaces over  $\mathbb{F}_b$ . By starting from the basis  $w_0, w_1, \dots, w_g$  of  $\mathcal{L}(G)$  and successively adding basis vectors at each step of the chain, we obtain for each  $n \in \mathbb{N}$  a basis

$$\{w_0, w_1, \dots, w_g, k_{i,1}, k_{i,2}, \dots, k_{i,ne_i}\} \quad (3.2)$$

of  $\mathcal{L}(G + nP_i)$ . We note that we then have

$$k_{i,j} \in \mathcal{L}(G + ((j-1)/e_i + 1)P_i) \quad \text{for } 1 \leq i \leq s \quad \text{and } j \geq 1. \quad (3.3)$$

By the Riemann-Roch theorem, there exists a local parameter  $z$  at  $P_\infty$ , e.g., with

$$\deg((z)_\infty) \leq 2g + e_1 \quad \text{for } z \in \mathcal{L}(G + P_1 - P_\infty) \setminus \mathcal{L}(G + P_1 - 2P_\infty). \quad (3.4)$$

For  $r \in \mathbb{N} \cup \{0\}$ , we put

$$z_r = \begin{cases} z^r & \text{if } r \notin \{n_0, n_1, \dots, n_g\}, \\ w_u & \text{if } r = n_u \text{ for some } u \in \{0, 1, \dots, g\}. \end{cases} \quad (3.5)$$

Note that in this case  $\nu_{P_\infty}(z_r) = r$  for all  $r \in \mathbb{N} \cup \{0\}$ . For  $1 \leq i \leq s$  and  $j \in \mathbb{N}$ , we have  $k_{i,j} \in \mathcal{L}(G + nP_i)$  for some  $n \in \mathbb{N}$  and also  $P_\infty \notin \text{supp}(G + nP_i)$ , hence  $\nu_{P_\infty}(k_j^{(i)}) \geq 0$ . Thus we have the local expansions

$$k_{i,j} = \sum_{r=0}^{\infty} a_{j,r}^{(i)} z_r \quad \text{for } 1 \leq i \leq s \quad \text{and } j \in \mathbb{N}, \quad (3.6)$$

where all coefficients  $a_{j,r}^{(i)} \in \mathbb{F}_b$ . For  $1 \leq i \leq s$  and  $j \in \mathbb{N}$ , we now define the sequences

$$\begin{aligned} \mathbf{c}_j^{(i)} &= (c_{j,0}^{(i)}, c_{j,1}^{(i)}, \dots) := (a_{j,n}^{(i)})_{n \in \mathbb{N}_0 \setminus \{n_0, \dots, n_g\}} \\ &= (\widehat{a_{j,n_0}^{(i)}} , a_{j,n_0+1}^{(i)}, \dots, \widehat{a_{j,n_1}^{(i)}} , a_{j,n_1+1}^{(i)}, \dots, \widehat{a_{j,n_g}^{(i)}} , a_{j,n_g+1}^{(i)}, \dots) \in \mathbb{F}_b^{\mathbb{N}}, \end{aligned} \quad (3.7)$$

where the hat indicates that the corresponding term is deleted. We define the matrices  $C^{(1)}, \dots, C^{(s)} \in \mathbb{F}_b^{\mathbb{N} \times \mathbb{N}}$  by

$$C^{(i)} = (\mathbf{c}_1^{(i)}, \mathbf{c}_2^{(i)}, \mathbf{c}_3^{(i)}, \dots)^\top \quad \text{for } 1 \leq i \leq s, \quad (3.8)$$

i.e., the vector  $\mathbf{c}_j^{(i)}$  is the  $j$ th row vector of  $C^{(i)}$  for  $1 \leq i \leq s$ .

**Theorem I** (see [DiPi, Theorem 8.11]). *With the above notations, we have that the matrices  $C^{(1)}, \dots, C^{(s)}$  given by (3.8) are generating matrices of the Xing-Niederreiter  $(t, s)$ -sequence  $(\mathbf{x}_n)_{n \geq 0}$  with  $t = g + e_0 - s$  and  $e_0 = e_1 + \dots + e_s$ .*

**Theorem 2.** *With the above notations,  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible, where  $d = g + e_0$ .*



(a) For  $s \geq 2$ ,  $e = e_1 \dots e_s$ ,  $m \geq 9(d+t)es^2\eta_1^{-1}$  and  $K_{d,t,s} = 4(d+t)(s-1)^2$ , we have

$$1 + \min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E_m^s} \max_{1 \leq N \leq b^m} \text{ND}^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w})_{0 \leq n < N}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} \eta_1^s m^s$$

with  $\eta_1 = (1 + \deg((z)_\infty))^{-1}$  (see (3.4)).

(b) Let  $s \geq 3$ ,  $\eta_2 \in (0, 1)$  and  $m \geq 8(d+t)e(s-1)^2\eta_2^{-1} + 2(1+2g+\eta_2 t)\eta_2^{-1}(1-\eta_2)^{-1}$ . Suppose that  $\min_{m/2-t \leq j \leq m} \nu_{P_\infty}(k_{i_0,j})/j \geq \eta_2$ , for some  $i_0 \in [1, s]$ . Then

$$\min_{\mathbf{w} \in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^m}) \geq 2^{-2} b^{-d} K_{d,t,s}^{-s+1} \eta_2^{s-1} m^{s-1}. \quad (3.9)$$

**2.3 Niederreiter-Özbudak nets** (see [DiPi, Section 8.2]). Let  $F/\mathbb{F}_b$  be an algebraic function field with full constant field  $\mathbb{F}_b$  and genus  $g = g(F/\mathbb{F}_b)$ . Let  $s \geq 2$ , and let  $P_1, \dots, P_s$  be  $s$  distinct places of  $F$  with degrees  $e_1, \dots, e_s$ . For  $1 \leq i \leq s$ , let  $\nu_{P_i}$  be the normalized discrete valuation of  $F$  corresponding to  $P_i$ , let  $t_i$  be a local parameter at  $P_i$ . Further, for each  $1 \leq i \leq s$ , let  $F_{P_i}$  be the residue class field of  $P_i$ , i.e.,  $F_{P_i} = O_{P_i}/P_i$ , and let  $\vartheta_i = (\vartheta_{i,1}, \dots, \vartheta_{i,e_i}) : F_{P_i} \rightarrow \mathbb{F}_b^{e_i}$  be an  $\mathbb{F}_b$ -linear vector space isomorphism. Let  $m > g + \sum_{i=1}^s (e_i - 1)$ . Choose an arbitrary divisor  $G$  of  $F/\mathbb{F}_b$  with  $\deg(G) = ms - m + g - 1$  and define  $a_i := \nu_{P_i}(G)$  for  $1 \leq i \leq s$ . For each  $1 \leq i \leq s$ , we define an  $F_b$ -linear map  $\theta_i : \mathcal{L}(G) \rightarrow \mathbb{F}_b^m$  on the Riemann-Roch space  $\mathcal{L}(G) = \{y \in F \setminus 0 : \text{div}(y) + G \geq 0\} \cup \{0\}$ . We fix  $i$  and repeat the following definitions related to  $\theta_i$  for each  $1 \leq i \leq s$ .

Note that for each  $f \in \mathcal{L}(G)$  we have  $\nu_{P_i}(f) \geq -a_i$ , and so the local expansion of  $f$  at  $P_i$  has the form

$$f = \sum_{j=-a_i}^{\infty} S_j(t_i, f) t_i^j, \quad \text{with } S_j(t_i, f) \in F_{P_i}, \quad j \geq -a_i. \quad (3.10)$$

We denote  $S_j(t_i, f)$  by  $f_{i,j}$ . Let  $m_i = \lfloor m/e_i \rfloor$  and  $r_i = m - e_i m_i$ . Note that  $0 \leq r_i < e_i$ . For  $f \in \mathcal{L}(G)$ , the image of  $f$  under  $\theta_i^{(G)}$ , for  $1 \leq i \leq s$ , is defined as

$$\theta_i^{(G)}(f) = (\theta_{i,1}(f), \dots, \theta_{i,m}(f)) := (\mathbf{0}_{r_i}, \vartheta_i(f_{i,-a_i+m_i-1}), \dots, \vartheta_i(f_{i,-a_i})) \in \mathbb{F}_b^m, \quad (3.11)$$

where we add the  $r_i$ -dimensional zero vector  $\mathbf{0}_{r_i} = (0, \dots, 0) \in \mathbb{F}_b^{r_i}$  in the beginning. Now we set

$$\theta^{(G)}(f) := (\theta_1^{(G)}(f), \dots, \theta_s^{(G)}(f)) \in \mathbb{F}_b^{ms}, \quad (3.12)$$

and define the  $\mathbb{F}_b$ -linear map

$$\theta^{(G)} : \mathcal{L}(G) \rightarrow \mathbb{F}_b^{ms}, \quad f \mapsto \theta^{(G)}(f).$$

The image of  $\theta^{(G)}$  is denoted by

$$\mathcal{N}_m = \mathcal{N}_m(P_1, \dots, P_s; G) := \{\theta^{(G)}(f) \in \mathbb{F}_b^{ms} \mid f \in \mathcal{L}(G)\}. \quad (3.13)$$

According to [DiPi, p.274],

$$\dim(\mathcal{N}_m) = \dim(\mathcal{L}(G)) \geq \deg(G) + 1 - g = ms - m \quad \text{for } m > g - s + e_1 + \dots + e_s.$$

Using the Riemann-Roch theorem, we get

$$\dim(\mathcal{N}_m) = ms - m \quad \text{for } m > g - s + e_1 + \dots + e_s, \quad s \geq 3. \quad (3.14)$$

Let  $\mathcal{N}_m^\perp = \mathcal{N}_m^\perp(P_1, \dots, P_s; G)$  be the dual space of  $\mathcal{N}_m(P_1, \dots, P_s; G)$  (see (2.21)). The space  $\mathcal{N}_m^\perp$  can be viewed as the row space of a suitable  $m \times ms$  matrix  $C$  over  $\mathbb{F}_b$ . Finally, we consider the digital net  $\mathcal{P}_1(\mathcal{N}_m^\perp) = \{\mathbf{x}_n(C) \mid n \in [0, b^m]\}$  with overall generating matrix  $C$  (see (2.19)).

Let  $\tilde{x}_i(h_i) = \sum_{j=1}^m \phi^{-1}(h_{i,j})b^{-j}$ , where  $h_i = (h_{i,1}, \dots, h_{i,m}) \in F_b^m$  ( $i = 1, \dots, s$ ) and let  $\tilde{\mathbf{x}}(\mathbf{h}) = (\tilde{x}_1(h_1), \dots, \tilde{x}_s(h_s))$  where  $\mathbf{h} = (h_1, \dots, h_s)$ . From (2.15), (2.16) and (2.20), we derive

$$\mathcal{P}_1 := \mathcal{P}_1(\mathcal{N}_m^\perp) = \{\tilde{\mathbf{x}}(\mathbf{h}) \mid \mathbf{h} \in \mathcal{N}_m^\perp(P_1, \dots, P_s; G)\}. \quad (3.15)$$

**Theorem J** (see [DiPi, Corollary 8.6]). *With the above notations, we have that  $\mathcal{P}_1$  is a  $(t, m, s)$ -net over  $\mathbb{F}_b$  with  $t = g + e_0 - s$  and  $e_0 = e_1 + \dots + e_s$ .*

To obtain a  $d$ -admissible net, we will consider also the following net:

$$\mathcal{P}_2 := \{(\{b^{r_1}z_1\}, \dots, \{b^{r_s}z_s\}) \mid \mathbf{z} = (z_1, \dots, z_s) \in \mathcal{P}_1\}. \quad (3.16)$$

Without loss of generality, let

$$e_s = \min_{1 \leq i \leq s} e_i. \quad (3.17)$$

**Theorem 3.** Let  $s \geq 3$ ,  $m_0 = 2^{2s+3}b^{d+t+s}(d+t)^s(s-1)^{2s-1}(g+e_0)e\eta^{-s+1}$  and  $\eta = (1 + \deg((t_s)_\infty))^{-1}$ . Then

$$\min_{\mathbf{w} \in E_m^s} \max_{1 \leq N \leq b^m} \text{ND}^*(\mathcal{P}_1 \oplus \mathbf{w}) \geq 2^{-2}b^{-d}K_{d,t,s}^{-s+1}\eta^{-s+1}m^{s-1}, \quad \text{for } m \geq m_0,$$

$\mathcal{P}_2$  is a  $d$ -admissible  $(t, m - r_0, s)$  net in base  $b$  with  $d = g + e_0$ ,  $t = g + e_0 - s$ , and

$$\min_{\mathbf{w} \in E_{m-r_0}^s} b^m \text{D}^*((\mathcal{P}_2 \oplus \mathbf{w})) \geq 2^{-2}b^{-d}K_{d,t,s}^{-s+1}\eta^{s-1}m^{-s+1}, \quad \text{for } m \geq m_0,$$

where  $\mathcal{P}_i \oplus \mathbf{w} := \{\mathbf{z} \oplus \mathbf{w} \mid \mathbf{z} \in \mathcal{P}_i\}$ .

**2.4 Halton-type sequence** (see [NiYe]). Let  $F/\mathbb{F}_b$  be an algebraic function field with full constant field  $\mathbb{F}_b$  and genus  $g = g(F/\mathbb{F}_b)$ . We assume that  $F/\mathbb{F}_b$  has at least one rational place, that is, a place of degree 1. Given a dimension  $s \geq 1$ , we choose  $s+1$  distinct places  $P_1, \dots, P_{s+1}$  of  $F$  with  $\deg(P_{s+1}) = 1$ . The degrees of the places  $P_1, \dots, P_s$  are arbitrary and we put  $e_i = \deg(P_i)$  for  $1 \leq i \leq s$ . Denote by  $O_F$  the holomorphy ring given by

$$O_F = \bigcap_{P \neq P_{s+1}} O_P,$$

where the intersection is extended over all places  $P \neq P_{s+1}$  of  $F$ , and  $O_P$  is the valuation ring of  $P$ . We arrange the elements of  $O_F$  into a sequence by using the fact that

$$O_F = \bigcup_{m=0}^{\infty} \mathcal{L}(mP_{s+1}).$$

The terms of this sequence are denoted by  $f_0, f_1, \dots$  and they are obtained as follows. Consider the chain

$$\mathcal{L}(0) \subseteq L(P_{s+1}) \subseteq L(2P_{s+1}) \subseteq \dots$$

of vector spaces over  $\mathbb{F}_b$ . At each step of this chain, the dimension either remains the same or increases by 1. From a certain point on, the dimension always increases by 1 according to the Riemann-Roch theorem. Thus we can construct a sequence  $v_0, v_1, \dots$  of elements of  $O_F$  such that

$$\{v_0, v_1, \dots, v_{\ell(mP_{s+1})-1}\} \tag{3.18}$$

is a  $\mathbb{F}_b$ -basis of  $\mathcal{L}(mP_{s+1})$ . For  $n \in \mathbb{N}$ , let

$$n = \sum_{r=0}^{\infty} a_r(n)b^r \quad \text{with all } a_r(n) \in Z_b$$

be the digit expansion of  $n$  in base  $b$ . Note that  $a_r(n) = 0$  for all sufficiently large  $r$ . We fix a bijection  $\phi : Z_b \rightarrow \mathbb{F}_b$  with  $\phi(0) = \bar{0}$ . Then we define

$$f_n = \sum_{r=0}^{\infty} \bar{a}_r(n) v_r \in O_F \quad \text{with} \quad \bar{a}_r(n) = \phi(a_r(n)) \quad \text{for } n = 0, 1, \dots \quad (3.19)$$

Note that the sum above is finite since for each  $n \in \mathbb{N}$  we have  $a_r(n) = 0$  for all sufficiently large  $r$ . By the Riemann-Roch theorem, we have

$$\{\tilde{f} \mid \tilde{f} \in \mathcal{L}((m+g-1)P_{s+1})\} = \{f_n \mid n \in [0, b^m]\} \quad \text{for } m \geq g. \quad (3.20)$$

For each  $i = 1, \dots, s$ , let  $\wp_i$  be the maximal ideal of  $O_F$  corresponding to  $P_i$ . Then the residue class field  $F_{P_i} := O_F/\wp_i$  has order  $b^{e_i}$  (see [St, Proposition 3.2.9]). We fix a bijection

$$\sigma_{P_i} : F_{P_i} \rightarrow Z_{b^{e_i}}. \quad (3.21)$$

For each  $i = 1, \dots, s$ , we can obtain a local parameter  $t_i \in O_F$  at  $\wp_i$ , by applying the Riemann-Roch theorem and choosing

$$t_i \in \mathcal{L}(kP_{s+1} - P_i) \setminus \mathcal{L}(kP_{s+1} - 2P_i) \quad (3.22)$$

for a suitably large integer  $k$ . We have a local expansion of  $f_n$  at  $\wp_i$  of the form

$$f_n = \sum_{j \geq 0} f_{n,j}^{(i)} t_i^j \quad \text{with all } f_{n,j}^{(i)} \in F_{P_i}, \quad n = 0, 1, \dots \quad (3.23)$$

We define the map  $\xi : O_F \rightarrow [0, 1]^s$  by

$$\xi(f_n) = \left( \sum_{j=0}^{\infty} \sigma_{P_1}(f_{n,j}^{(1)}) b^{-e_1(j+1)}, \dots, \sum_{j=0}^{\infty} \sigma_{P_s}(f_{n,j}^{(s)}) (b^{-e_s(j+1)}) \right). \quad (3.24)$$

Now we define the sequence  $\mathbf{x}_0, \mathbf{x}_1, \dots$  of points in  $[0, 1]^s$  by

$$\mathbf{x}_n = \xi(f_n) \quad \text{for } n = 0, 1, \dots \quad (3.25)$$

From [NiYe, Theorem 1], we get the following theorem :

**Theorem K.** *With the notation as above, we have that  $(\mathbf{x}_n)_{n \geq 0}$  is a  $(t, s)$ -sequence in base  $b$  with  $t = g + e_0 - s$  and  $e_0 = e_1 + \dots + e_s$ .*

By Lemma 17,  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible with  $d = g + e_0$ . Using [Le4, Theorem 2], we get

$$1 + \max_{1 \leq N \leq b^m} ND^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w})_{0 \leq n < N}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} m^s \quad (3.26)$$

for some  $Q \in [0, b^m)$  and  $\mathbf{w} \in E_m^s$ .

In order to obtain (3.26) for every  $Q$  and  $\mathbf{w}$ , we choose a specific sequence  $v_0, v_1, \dots$  as follows. Let

$$t_{s+1} \in \mathcal{L}([(2g+1)/e_1] + 1)P_1 - P_{s+1} \setminus \mathcal{L}([(2g+1)/e_1] + 1)P_1 - 2P_{s+1}.$$

It is easy to see that

$$\nu_{P_{s+1}}(t_{s+1}) = 1, \quad \nu_{P_i}(t_{s+1}) \geq 0, \quad i \in [2, s] \quad \text{and} \quad \deg((t_{s+1})_\infty) \leq 2g + e_1 + 1. \quad (3.27)$$

By (3.18) and the Riemann-Roch theorem, we have  $\nu_{P_{s+1}}(v_i) = -i - g$  for  $i \geq g$ . Hence

$$v_i = \sum_{j \leq i+g} v_{i,j} t_{s+1}^{-j} \quad \text{with} \quad \text{all} \quad v_{i,j} \in \mathbb{F}_b, \quad v_{i,i+g} \neq 0, \quad i \geq g. \quad (3.28)$$

Using the orthogonalization procedure, we can construct a sequence  $v_0, v_1, \dots$  such that  $\{v_0, v_1, \dots, v_{\ell(mP_{s+1})-1}\}$  is a  $\mathbb{F}_b$ -basis of  $\mathcal{L}(mP_{s+1})$ ,

$$v_{i,i+g} = 1, \quad \text{and} \quad v_{i,j+g} = 0 \quad \text{for} \quad j \in [g, i], \quad i \geq g. \quad (3.29)$$

Subsequently, we will use just this sequence.

**Theorem 4.** *With the above notations,  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible, where  $d = g + e_0$ .*

(a) *For  $s \geq 2$ ,  $m \geq 2^{2s+3} b^{d+t+s+1} (d+t)^{s+1} s^{2s} e(g+1)(e_0+s) \eta_1^{-s}$  and  $\eta_1 = (1 + \deg((t_{s+1})_\infty))^{-1}$ , we have*

$$1 + \min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E_m^s} \max_{1 \leq N \leq b^m} ND^*((\mathbf{x}_{n \oplus Q} \oplus \mathbf{w})_{0 \leq n < N}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} \eta_1^s m^s. \quad (3.30)$$

(b) *Let  $s \geq 3$ ,  $m \geq 2^{2s+3} b^{d+t+s} (d+t)^s (s-1)^{2s-1} (g+e_0) e \eta_2^{-s+1}$ ,  $e_s = \min_{1 \leq i \leq s} e_i$  and  $\eta_2 = (1 + \deg((t_s)_\infty))^{-1}$ . Then*

$$\min_{\mathbf{w} \in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^m}) \geq 2^{-2} b^{-d} K_{d,t,s}^{-s+1} \eta_2^{s-1} m^{s-1}. \quad (3.31)$$

### 2.5. Niederreiter-Xing sequence.

Let  $F/\mathbb{F}_b$  be an algebraic function field with full constant field  $\mathbb{F}_b$  and genus  $g = g(F/\mathbb{F}_b)$ . Assume that  $F/\mathbb{F}_b$  has at least  $s + 1$  rational places. Let  $P_1, \dots, P_{s+1}$  be  $s + 1$  distinct rational places of  $F$ . Let  $G_m = m(P_1 + \dots + P_s) - (m - g + 1)P_{s+1}$ , and let  $t_i$  be a local parameter at  $P_i$ ,  $1 \leq i \leq s + 1$ . For any  $f \in \mathcal{L}(G_m)$  we have  $\nu_{P_i}(f) \geq m$ , and so the local expansion of  $f$  at  $P_i$  has the form

$$f = \sum_{j=-m}^{\infty} f_{i,j} t_i^j, \quad \text{with } f_{i,j} \in \mathbb{F}_b, \quad j \geq -m, \quad 1 \leq i \leq s.$$

For  $1 \leq i \leq s$ , we define the  $\mathbb{F}_b$ -linear map  $\psi_{m,i}(f) : \mathcal{L}(G_m) \rightarrow \mathbb{F}_b^m$  by

$$\psi_{m,i}(f) = (f_{i,-1}, \dots, f_{i,-m}) \in \mathbb{F}_b^m, \quad \text{for } f \in \mathcal{L}(G_m).$$

Let

$$\mathcal{M}_m = \mathcal{M}_m(P_1, \dots, P_s; G_m) := \{(\psi_{m,1}(f), \dots, \psi_{m,s}(f)) \in \mathbb{F}_b^{ms} \mid f \in \mathcal{L}(G_m)\}. \quad (3.32)$$

Let  $C^{(1)}, \dots, C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$  be the generating matrices of a digital sequence  $\mathbf{x}_n(C)_{n \geq 0}$ , and let  $(\mathcal{C}_m)_{m \geq 1}$  be the associated sequence of row spaces of overall generating matrices  $[C]_m$ ,  $m = 1, 2, \dots$  (see (2.19)).

**Theorem L.** (see [DiPi, Theorem 7.26 and Theorem 8.9]) *There exist matrices  $C^{(1)}, \dots, C^{(s)}$  such that  $\mathbf{x}_n(C)_{n \geq 0}$  is a digital  $(t, s)$ -sequence with  $t = g$  and  $\mathcal{C}_m = \mathcal{M}_m^\perp(P_1, \dots, P_s; G_m)$  for  $m \geq g + 1$ ,  $s \geq 2$ .*

According to [DiNi, p.411] and [DiPi, p.275], the construction of digital sequences of Niederreiter and Xing [NiXi] can be achieved by using the above approach. We propose the following way to get  $\mathbf{x}_n(C)_{n \geq 0}$ .

We consider the  $H$ -differential  $dt_{s+1}$ . Let  $\omega$  be the corresponding Weil differential,  $\text{div}(\omega)$  the divisor of  $\omega$ , and  $W := \text{div}(dt_{s+1}) = \text{div}(\omega)$ . By (2.5), we have  $\deg(W) = 2g - 2$ . Similarly to (3.18)-(3.29), we can construct a sequence  $\dot{v}_0, \dot{v}_1, \dots$  of elements of  $F$  such that  $\{\dot{v}_0, \dot{v}_1, \dots, \dot{v}_{\ell((m-g+1)P_{s+1}+W)-1}\}$  is a  $\mathbb{F}_b$ -basis of  $L_m := \mathcal{L}((m - g + 1)P_{s+1} + W)$  and

$$\dot{v}_r \in L_{r+1} \setminus L_r, \quad \nu_{P_{s+1}}(\dot{v}_r) = -r + g - 2, \quad r \geq g, \quad \text{and } \dot{v}_{r,r+2-g} = 1, \quad \dot{v}_{r,j} = 0 \quad (3.33)$$

for  $2 \leq j < r + 2 - g$ , where

$$\dot{v}_r := \sum_{j \leq r-g+2} \dot{v}_{r,j} t_{s+1}^{-j} \quad \text{for } \dot{v}_{r,j} \in \mathbb{F}_b \text{ and } r \geq g.$$

According to Proposition A, we have that there exists  $\tau_i \in F$  ( $1 \leq i \leq s$ ), such that  $dt_{s+1} = \tau_i dt_i$  for  $1 \leq i \leq s$ .

Bearing in mind (2.4), (2.6) and (3.33), we get

$$\nu_{P_i}(\dot{v}_r \tau_i) = \nu_{P_i}(\dot{v}_r \tau_i dt_i) = \nu_{P_i}(\dot{v}_r dt_{s+1}) \geq \nu_{P_i}(\text{div}(dt_{s+1}) - W) = 0, \quad 1 \leq i \leq s, \quad r \geq 0.$$

We consider the following local expansions

$$\dot{v}_r \tau_i := \sum_{j=0}^{\infty} \dot{c}_{j,r}^{(i)} t_i^j, \quad \text{where all } \dot{c}_{j,r}^{(i)} \in \mathbb{F}_b, \quad 1 \leq i \leq s, \quad j \geq 0. \quad (3.34)$$

Now let  $\dot{C}^{(i)} = (\dot{c}_{j,r}^{(i)})_{j,r \geq 0}$ ,  $1 \leq i \leq s$ , and let  $\dot{C}_m$  be the row space of overall generating matrix  $[\dot{C}]_m$  (see (2.19)).

**Theorem 5.** *With the above notations,  $\mathbf{x}_n(\dot{C})_{n \geq 0}$  is a digital  $d$ -admissible  $(t, s)$  sequence, satisfying the bounds (3.30) and (3.31), with  $d = g + s$ ,  $t = g$ , and  $\dot{C}_m = \mathcal{M}_m^\perp(P_1, \dots, P_s; G_m)$  for all  $m \geq g + 1$ .*

**2.6 General  $d$ -admissible digital  $(t, s)$  sequences.** In [KrLaPi], discrepancy bounds for index-transformed uniformly distributed sequences was studied. In this subsection, we consider a lower bound of such a sequences.

Let  $s \geq 2$ ,  $d \geq 1$ ,  $t \geq 0$ ,  $d_0 = d + t$  and  $m_k = s^2 d_0 (2^{2k+2} - 1)$  for  $k = 1, 2, \dots$ . Let  $C^{(s+1)} = (c_{i,j}^{(s+1)})_{i,j \geq 1}$  be a  $\mathbb{N} \times \mathbb{N}$  matrix over  $\mathbb{F}_b$ , and let  $[C^{(s+1)}]_{m_k}$  be a nonsingular matrix,  $k = 1, 2, \dots$ . For  $n \in [0, b^{m_k})$ , let  $\mathbf{h}_k(n) = (h_{k,1}(n), \dots, h_{k,m_k}(n)) = \mathbf{n}[C^{(s+1)}]_{m_k}^\top$  and  $h_k(n) = \sum_{j=1}^m \phi^{-1}(h_{k,j}(n)) b^{j-1}$  ( $k \geq 1$ ). We have  $h_k(l) \neq h_k(n)$  for  $l \neq n$ ,  $l, n \in [0, b^{m_k})$ . Let  $h_k^{-1}(h_k(n)) = n$  for  $n \in [0, b^{m_k})$ . It is easy to see that  $h_k^{-1}$  is a bijection from  $[0, b^{m_k})$  to  $[0, b^{m_k})$  ( $k = 1, 2, \dots$ ).

**Theorem 6.** *Let  $(\mathbf{x}_n)_{n \geq 0}$  be a digital  $d$ -admissible  $(t, s)$  sequence in base  $b$ . Then there exists a matrix  $\bar{C}^{(s+1)}$  and a sequence  $(h^{-1}(n))_{n \geq 0}$  such that  $[C^{(s+1)}]_{m_k}$  is nonsingular,  $h^{-1}(n) = h_l^{-1}(n) = h_k^{-1}(n)$  for  $n \in [0, b^{m_k})$  ( $l > k$ ,  $k = 1, 2, \dots$ ),  $(\mathbf{x}_{h^{-1}(n)})_{n \geq 0}$  a  $d$ -admissible  $(t, s)$  sequence in base  $b$ , and*

$$1 + \min_{0 \leq Q < b^{m_k}, \mathbf{w} \in E_{m_k}^s} \max_{1 \leq N \leq b^{m_k}} ND^*((\mathbf{x}_{h^{-1}(n) \oplus Q \oplus \mathbf{w}})_{0 \leq n < N}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} m_k^s, \quad k \geq 1.$$

**Remark 1.** Halton-type sequences were introduced in [Te1] for the case of rational function fields over finite fields. Generalizations to the general case of algebraic function field were obtained in [Le1] and [NiYe]. The constructions in [Le1] and [NiYe] are similar. The difference is that the construction in [NiYe] is more simple, but the construction in [Le1] is somewhat more general.

**Remark 2.** We note that all explicit constructions of this article are expressed in terms of the residue of a differential and are similar to the Halton construction (see, e.g., (4.6), (4.28), (4.62) and (4.113)-(4.121)). The earlier constructions of  $(t, s)$ -sequences using differentials, see e.g. [MaNi].

**Remark 3.** A lattice  $\Gamma \subset (\mathbb{F}_b((x)))^{s+1}$  is  $d$ -admissible if

$$\inf_{(\gamma_1, \dots, \gamma_{s+1}) \in \Gamma \setminus \{0\}} \prod_{i=1}^{s+1} b^{-\nu_\infty(\gamma_i)} \geq b^{-d}.$$

A lattice  $\Gamma \subset (\mathbb{F}_b((x)))^{s+1}$  is said to be **admissible** if  $\Gamma$  is  $d$ -admissible with some  $d > 0$ . In [Le1, Theorem 3.2], we proved the following analog of the main theorem of the duality theory (see, [DiPi, Section 7], [NiPi] and [Skr]): if a unimodular lattice  $\Gamma$  is  $d$ -admissible, then from the dual lattice  $\Gamma^\perp$  we can get a  $(t, s)$  sequence with  $t = d - s$ . It is easy to prove (see, e.g., [Le5]) that a lattice  $\Gamma$  is admissible if and only if the dual lattice  $\Gamma^\perp$  is admissible. In [Le4] and in this paper we consider a more general object. We consider nets in  $[0, 1]^s$  having simultaneously both  $(t, m, s)$  properties and  $d$ -admissible properties. The  $d$ -admissible properties have a direct connection to the notion of the weight in the duality theory (see Definition 5, Definition 8 and Definition 9). Thus we can consider this paper as a part of the duality theory.

## 4 Proof of theorems.

### 4.1 Generalized Niederreiter sequence. Proof of Theorem 1

Using [Le4, Lemma 2] and [Te3, Theorem 1], we obtain that  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible with  $d = e_0$ .

We apply Corollary 3 with  $B'_i = \emptyset$ ,  $1 \leq i \leq s + 1$ ,  $B = 0$ ,  $\hat{e} = e = e_1 e_2 \cdots e_s$ ,  $d_0 = d + t$ ,  $\epsilon = \eta_1 (2s d_0 e)^{-1}$  and  $\eta_1 = s/(s + 1)$ . In order to prove the first assertion



in Theorem 1, it is sufficient to verify that

$$\Lambda_1 = \mathbb{F}_b^{(s+1)d_0e[m\epsilon]}, \quad \text{for} \quad m \geq 9(d+t)es(s+1), \quad (4.1)$$

where

$$\Lambda_1 = \{(y_{n,1}^{(1)}, \dots, y_{n,d_1}^{(1)}, \dots, y_{n,1}^{(s)}, \dots, y_{n,d_s}^{(s)}, \bar{a}_{d_{s+1},1}(n), \dots, \bar{a}_{d_{s+1},2}(n)) \mid n \in [0, b^m)\}$$

with

$$d_i = \dot{m}_i = d_0e[m\epsilon] \quad (1 \leq i \leq s), \quad d_{s+1,1} = \ddot{m}_{s+1} + 1 := t + (s-1)d_0e[m\epsilon], \quad (4.2)$$

$$d_{s+1,2} = \dot{m}_{s+1} := t - 1 + sd_0e[m\epsilon], \text{ and } n = \sum_{0 \leq j \leq m-1} a_j(n)b^j.$$

Suppose that (4.1) is not true. Then there exists  $b_{i,j} \in \mathbb{F}_b$  ( $i, j \geq 1$ ) such that

$$\sum_{i=1}^s \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0 \quad (4.3)$$

and

$$\sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} y_{n,j}^{(i)} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} \bar{a}_j(n) = 0 \quad \text{for all } n \in [0, b^m). \quad (4.4)$$

From (2.14) and (3.1), we have

$$y_{n,j}^{(i)} = \sum_{r=0}^{m-1} c_{j,r}^{(i)} \bar{a}_r(n),$$

with

$$c_{j,r}^{(i)} = a^{(i)}(Q+1, k, r) \in \mathbb{F}_b, \quad j-1 = Qe_i + k, \quad 0 \leq k < e_i, \quad (4.5)$$

$Q = Q(i, j)$ ,  $k = k(i, j)$ , where  $a^{(i)}(j, k, r)$  are defined from the expansions

$$\frac{y_{i,j,k}(x)}{p_i(x)^j} = \sum_{r \geq 0} a^{(i)}(j, k, r) x^{-r-1}.$$

We consider the field  $F = \mathbb{F}_b(x)$ , the valuation  $\nu_\infty$  (see (2.1)) and the place  $P_\infty = \text{div}(x^{-1})$ . By (2.8), we get

$$a^{(i)}(j, k, r) = \text{Res}_{P_\infty, x^{-1}} (y_{i,j,k}(x) p_i(x)^{-j} x^{r+2}).$$

Hence

$$y_{n,j}^{(i)} = \text{Res}_{P_\infty, x^{-1}} \left( \frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} \sum_{r=0}^{m-1} \bar{a}_r(n) x^{r+2} \right) = \text{Res}_{P_\infty, x^{-1}} \left( \frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} n(x) \right) \quad (4.6)$$

with  $n(x) = \sum_{j=0}^{m-1} \bar{a}_j(n) x^{j+2}$  for all  $j \in [1, d_i]$ ,  $i \in [1, s]$ .

We have  $\bar{a}_j(n) = \text{Res}_{P_\infty, x^{-1}} (n(x) x^{-j-1})$ . From (4.4), we derive

$$\text{Res}_{P_\infty, x^{-1}} (n(x) \alpha) = 0 \text{ with } \alpha = \sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} \frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} x^{-j-1} \quad (4.7)$$

for all  $n \in [0, b^m]$ . Consider the local expansion

$$\alpha = \sum_{r=0}^{\infty} \varphi_r x^{-r-1} \quad \text{with} \quad \varphi_r \in \mathbb{F}_b, \quad r \geq 0.$$

Applying (2.12) and (4.7), we derive

$$\begin{aligned} \text{Res}_{P_\infty, x^{-1}} (n(x) \alpha) &= \text{Res}_{P_\infty, x^{-1}} \left( \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) x^{\mu+2} \sum_{r=0}^{\infty} \varphi_r x^{-r-1} \right) = \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) \varphi_r \\ &\times \text{Res}_{P_\infty, x^{-1}} (x^{\mu+2-r-1}) = \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) \varphi_r \delta_{\mu,r} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \varphi_\mu = 0 \end{aligned}$$

for all  $n \in [0, b^m]$ . Hence

$$\varphi_r = 0 \quad \text{for} \quad r \in [0, m-1] \quad \text{and} \quad \nu_\infty(\alpha) \geq m. \quad (4.8)$$

According to (4.5), we obtain

$$Q(i, j) + 1 \leq Q(i, d_i) + 1 \leq [(d_i - 1)/e_i] + 1 = d_i/e_i \text{ for } j \in [1, d_i], i \in [1, s].$$

By (4.7), we get

$$\alpha \in \mathcal{L}(G_1) \quad \text{with} \quad G_1 = \sum_{i=1}^s d_i/e_i \text{div}(p_i(x)) + (d_{s+1,2} + 1) \text{div}(x) - mP_\infty. \quad (4.9)$$

From (4.1) and (4.2), we have for  $m \geq 2t + 8(d+t)es(s+1)$

$$\begin{aligned} \deg(G_1) &= \sum_{i=1}^s d_i + d_{s+1,2} + 1 - m = sd_0e[m\epsilon] + t - 1 + sd_0e[m\epsilon] + 1 - m \\ &\leq t - m(1 - 2sd_0e\epsilon) = t - m(1 - \eta_1) = t - m/(s+1) < 0. \end{aligned}$$

Hence  $\alpha = 0$ .

Let  $\text{g.c.d.}(x, p_j(x)) = 1$  for all  $j \neq i$  with some  $i \in [1, s]$ . For example, let  $i = 1$ , and let  $p_1(x) = x^{e_{1,1}}\dot{p}_1(x)$  with  $e_{1,2} = \deg(\dot{p}_1(x))$ ,  $e_1 = e_{1,1} + e_{1,2}$ ,  $e_{1,1} \geq 0$ ,  $\text{g.c.d.}(x, \dot{p}_1(x)) = 1$ . According to (4.7), we get  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ , where

$$\begin{aligned} \alpha_1 &= \sum_{i=2}^s \sum_{j=1}^{d_i} b_{i,j} \frac{y_{i,Q(i,j)+1,k(1,j)}(x)}{p_i(x)^{Q(i,j)+1}}, & \alpha_2 &= \sum_{j=1}^{d_1} b_{1,j} \frac{\ddot{y}_{1,Q(1,j)+1,k(1,j)}(x)}{\dot{p}_1(x)^{Q(1,j)+1}} \\ \text{and } \alpha_3 &= \sum_{j=1}^{d_1} b_{1,j} \frac{\dot{y}_{1,Q(1,j)+1,k(1,j)}(x)}{x^{e_{1,1}(Q(1,j)+1)}} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} \frac{b_{s+1,j}}{x^{j+1}} \end{aligned}$$

with some polynomials  $\dot{y}_{1,j,k}(x)$  and  $\ddot{y}_{1,j,k}(x)$ .

Using (4.2), we obtain for  $s \geq 2$  and  $j \in [1, d_1]$  that

$$d_{s+1,1} + 1 = t + 1 + (s-1)d_0e[m\epsilon] > d_0e[m\epsilon] = d_1 \geq e_{1,1}d_1/e_1 \geq e_{1,1}\deg(Q(1, d_1) + 1).$$

We have that the polynomials  $p_2, \dots, p_s, \dot{p}_1$  and  $x$  are pairwise coprime over  $\mathbb{F}_b$ . By the uniqueness of the partial fraction decomposition of a rational function, we have that  $\alpha_3 = 0$  and  $b_{s+1,j} = 0$  for all  $j \in [d_{s+1,1}, d_{s+1,2}]$ .

Bearing in mind that  $p_1, \dots, p_s$  are pairwise coprime polynomials over  $\mathbb{F}_b$ , we obtain from [Te3, p.242] or [Te2, p. 166,167] that  $b_{i,j} = 0$  for all  $j \in [1, d_i]$  and  $i \in [1, s]$ .

By (4.3), we have the contradiction. Hence assertion (4.1) is true. Thus the first assertion in Theorem 1 is proved.

Now consider the second assertion in Theorem 1:

Let, for example,  $i_0 = s$ , i.e.

$$\min_{m/2-t \leq j \leq s, 0 \leq k < e_s} (1 - \deg(y_{s,j,k}(x))j^{-1}e_s^{-1}) \geq \eta_2. \quad (4.10)$$

We apply Corollary 2 with  $\dot{s} = s \geq 3$ ,  $B_i = \emptyset$ ,  $1 \leq i \leq s$ ,  $B = 0$ ,  $\tilde{r} = 0$ ,  $m = \tilde{m}$ ,  $d_0 = d + t$ ,  $\hat{e} = e = e_1e_2 \cdots e_s$ ,  $\epsilon = \eta_2(2(s-1)d_0e)^{-1}$ . In order to prove the second assertion in Theorem 1, it is sufficient to verify that

$$\Lambda_2 = \mathbb{F}_b^{sd_0e[m\epsilon]} \quad \text{for} \quad m \geq 8(d+t)e(s-1)^2\eta_2^{-1} + 2(1+t)\eta_2^{-1}(1-\eta_2)^{-1}, \quad (4.11)$$

where

$$\Lambda_2 = \{(y_{n,1}^{(1)}, \dots, y_{n,d_1}^{(1)}, \dots, y_{n,1}^{(s-1)}, \dots, y_{n,d_{s-1}}^{(s-1)}, y_{n,d_{s,1}}^{(s)}, \dots, y_{n,d_{s,2}}^{(s)}) \mid n \in [0, b^m)\},$$

with

$$d_i = \dot{m}_i = d_0 e[m\epsilon], \quad i \in [1, s), \quad d_{s,1} = \ddot{m}_s + 1 := m - t + 1 - (s-1)d_0 e[m\epsilon] \quad (4.12)$$

and  $d_{s,2} = \dot{m}_s := m - t - (s-2)d_0 e[m\epsilon]$ .

Suppose that (4.11) is not true. Then there exists  $b_{i,j} \in \mathbb{F}_b$  ( $i, j \geq 1$ ) such that

$$\sum_{i=1}^{s-1} \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s,1}}^{d_{s,2}} |b_{s,j}| > 0 \quad (4.13)$$

and

$$\sum_{i=1}^{s-1} \sum_{j=1}^{d_i} b_{i,j} y_{n,j}^{(i)} + \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} y_{n,j}^{(s)} = 0 \quad \text{for all } n \in [0, b^m). \quad (4.14)$$

Similarly to (4.7), we have

$$\text{Res}_{P_\infty, x^{-1}}(n(x)\alpha) = 0 \quad \text{for all } n \in [0, b^m), \quad \text{with } \alpha = \alpha_1 + \alpha_2,$$

where

$$\alpha_1 = \sum_{i=1}^{s-1} \sum_{j=1}^{d_i} b_{i,j} \frac{y_{i,Q(i,j)+1,k(i,j)}(x)}{p_i(x)^{Q(i,j)+1}} \quad \text{and} \quad \alpha_2 = \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} \frac{y_{s,Q(s,j)+1,k(s,j)}(x)}{p_s(x)^{Q(s,j)+1}}. \quad (4.15)$$

Consider the local expansions

$$\alpha_1 = \sum_{r=0}^{\infty} \varphi_{1,r} x^{-r-1} \quad \text{and} \quad \alpha_2 = \sum_{r=0}^{\infty} \varphi_{2,r} x^{-r-1} \quad \text{with } \varphi_{i,r} \in \mathbb{F}_b \quad i = 1, 2, \quad r \geq 0.$$

Analogously to (4.8), we obtain from (4.14)

$$\varphi_{1,r} + \varphi_{2,r} = 0 \quad \text{for all } r \in [0, m-1]. \quad (4.16)$$

Taking into account that  $j \leq (Q(s,j)+1)e_s$  and  $d_{s,1} \geq m/2 - t$ , we get from (2.1) and (4.10) that

$$\nu_\infty \left( \frac{y_{s,Q(s,j)+1,k(s,j)}(x)}{p_s(x)^{Q(s,j)+1}} \right) = (Q(s,j)+1)e_s - \deg(y_{s,Q(s,j)+1,k(s,j)}(x)) =$$

$$(Q(s, j) + 1) \left( 1 - \frac{\deg(y_{s, Q(s, j) + 1, k(s, j)}(x))}{(Q(s, j) + 1)e_s} \right) e_s \geq (Q(s, j) + 1)e_s \eta_2 \geq \eta_2 j, \quad j \geq d_{s, 1}.$$

Applying (4.15)-(4.16), we have  $\varphi_{2, r} = 0$  for  $r < [\eta_2 d_{s, 1}]$ . Therefore  $\varphi_{1, r} = 0$  for  $r < [\eta_2 d_{s, 1}]$ . Hence

$$\nu_\infty(\alpha_1) \geq [\eta_2 d_{s, 1}].$$

Similarly to (4.9), we obtain

$$\alpha_1 \in \mathcal{L}(G_2) \quad \text{with} \quad G_2 = \sum_{i=1}^{s-1} d_i / e_i \operatorname{div}(p_i(x)) - [\eta_2 d_{s, 1}] P_\infty.$$

From (4.11) and (4.12), we have that  $m > 2(1+t)\eta_2^{-1}(1-\eta_2)^{-1}$  and

$$\begin{aligned} \deg(G_2) &= \sum_{i=1}^{s-1} d_i - [d_{s, 1} \eta_2] = (s-1)d_0 e[m\epsilon] - [(m-t+1-(s-1)d_0 e[m\epsilon])\eta_2] \\ &\leq (s-1)d_0 e[m\epsilon] - (m-t-(s-1)d_0 e[m\epsilon])\eta_2 + 1 = (1+\eta_2)(s-1)d_0 e[m\epsilon] \\ &\quad - m\eta_2 + 1 + t \leq m((1+\eta_2)((s-1)d_0 e\epsilon - \eta_2) + 1 + t) \\ &= m\eta_2((1+\eta_2)/2 - 1) + 1 + t = 1 + t - m\eta_2(1-\eta_2)/2 < 0. \end{aligned}$$

Hence  $\alpha_1 = 0$  and  $\varphi_{1, r} = 0$  for  $r \geq 0$ .

Using [Te3, p.242] or [Te2, p. 166, 167], we get  $b_{i, j} = 0$  for all  $j \in [1, d_i]$  and  $i \in [1, s-1]$ .

According to (4.16), we have  $\varphi_{2, r} = 0$  for  $r \in [0, m-1]$ . Thus  $\nu_\infty(\alpha_2) \geq m$ .

From (4.15), we obtain

$$\alpha_2 \in \mathcal{L}(G_3) \quad \text{with} \quad G_3 = [d_{s, 2}/e_s + 1] \operatorname{div}(p_s(x)) - mP_\infty.$$

Applying (4.1) and (4.2), we derive for  $m > 2/\epsilon$  and  $s \geq 3$

$$\deg(G_3) \leq m - t - (s-2)d_0 e[m\epsilon] + e_s - m < 0.$$

Hence  $\alpha_2 = 0$ .

By the uniqueness of the partial fraction decomposition of a rational function, we have from (4.15) that  $b_{s+1, j} = 0$  for all  $j \in [d_{s, 1}, d_{s, 2}]$ .

By (4.13), we have a contradiction. Thus assertion (4.11) is true. Therefore Theorem 1 is proved.  $\square$

## 4.2 Xing-Niederreiter sequence. Proof of Theorem 2

**Lemma 3.** *Let  $P \in \mathbb{P}_F$ ,  $t$  be a local parameter of  $P$  over  $F$ ,  $k_j \in F$ ,  $\nu_P(k_j) = j$  ( $j = 0, 1, \dots$ ). Then there exists  $k_j^\perp \in F$  with  $\nu_P(k_j^\perp) = -j$  ( $j = 1, 2, \dots$ ), such that*

$$S_{-1}(t, k_{j_1} k_{j_2+1}^\perp) = \delta_{j_1, j_2} \quad \text{for} \quad j_1, j_2 \geq 0. \quad (4.17)$$

**Proof.** Let  $k_1^\perp = (tk_0)^{-1}$ . We see  $\nu_P(k_j k_1^\perp) \geq 0$  for  $j \geq 1$ . Using (2.2) and (2.12), we get that (4.17) is true for  $j_2 = 0$ . Suppose that the assertion of the lemma is true for  $0 \leq j_2 \leq j_0 - 1$ ,  $j_0 \geq 1$ . We take

$$k_{j_0+1}^\perp = \sum_{\mu=1}^{j_0} \rho_{\mu, j_0} k_\mu^\perp + (tk_{j_0})^{-1}, \quad \text{where} \quad \rho_{\mu, j_0} = S_{-1}(t, k_{\mu-1} (tk_{j_0})^{-1}). \quad (4.18)$$

We see that  $\nu_P(k_{j_0+1}^\perp) = -j_0 - 1$ . By the condition of the lemma and the assumption of the induction, we have  $\nu_P(k_{j_1} k_{j_0+1}^\perp) \geq 0$  for  $j_1 > j_0$  and

$$S_{-1}(t, k_{j_1} k_{j_0+1}^\perp) = \delta_{j_1, j_0} \quad \text{for} \quad j_1 \geq j_0. \quad (4.19)$$

Now consider the case  $j_1 \in [0, j_0]$ . Applying (4.18), we derive

$$S_{-1}(t, k_{j_1} k_{j_0+1}^\perp) = \sum_{\mu=1}^{j_0} \rho_{\mu, j_0} S_{-1}(t, k_{j_1} k_\mu^\perp) + S_{-1}(t, k_{j_1} (tk_{j_0})^{-1}).$$

Using (2.12), (4.18) and the assumption of the induction, we get

$$S_{-1}(t, k_{j_1} k_{j_0+1}^\perp) = \sum_{\mu=1}^{j_0} \rho_{\mu, j_0} \delta_{j_1, \mu-1} + S_{-1}(t, k_{j_1} (tk_{j_0})^{-1}) = \rho_{j_1+1, j_0} - \rho_{j_1+1, j_0} = 0.$$

Hence (4.19) is true for all  $j_1 \geq 0$ . By induction, Lemma 3 is proved.  $\square$

**Lemma 4.**  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible with  $d = g + e_0$ , where  $e_0 = e_1 + \dots + e_s$ .

**Proof.** Consider Definition 5. Taking into account that  $(\mathbf{x}_n)_{n \geq 0}$  is a digital sequence in base  $b$ , we can take  $k = 0$ . Suppose that the assertion of the lemma is not true. By (1.4), there exists  $\tilde{n} > 0$  such that  $\|\tilde{n}\|_b \|\mathbf{x}_{\tilde{n}}\|_b < b^{-d} = b^{-g-e_0}$ .

Let  $d_i = \dot{d}_i e_i + \ddot{d}_i$  with  $0 \leq \ddot{d}_i < e_i$ ,  $1 \leq i \leq s$ ,  $\|\tilde{n}\|_b = b^{m-1}$  and let  $\left\| \mathbf{x}_{\tilde{n}}^{(i)} \right\|_b = b^{-d_i-1}$ ,  $1 \leq i \leq s$ . Hence  $\tilde{n} \in [b^{m-1}, b^m)$ ,  $x_{\tilde{n}, d_i+1}^{(i)} \neq 0$ ,

$$x_{\tilde{n}, j}^{(i)} = 0 \quad \text{for all} \quad j \in [1, d_i], \quad i \in [1, s] \quad \text{and} \quad \sum_{i=1}^s (d_i + 1) - m \geq d = g + e_0.$$

By (2.14), we have

$$y_{\tilde{n},j}^{(i)} = 0 \quad \text{for all } j \in [1, \dot{d}_i e_i], \quad i \in [1, s] \quad \text{with} \quad \sum_{i=1}^s \dot{d}_i e_i \geq m + g. \quad (4.20)$$

Let

$$\{\dot{n}_0, \dots, \dot{n}_{g-1}\} = \{0, 1, \dots, 2g\} \setminus \{n_0, n_1, \dots, n_g\} \quad \text{and} \quad \dot{n}_i = g + i + 1 \text{ for } i \geq g. \quad (4.21)$$

Let  $n = \sum_{i=0}^{m-1} a_i(n) b^i$  with  $a_i(n) \in Z_b$  ( $i = 0, 1, \dots$ ), and let  $\bar{a}_i(n) = \phi(a_i(n))$  ( $i = 0, 1, \dots$ ) (see (2.13)). From (2.14), (3.6) and (3.7), we get

$$y_{n,j}^{(i)} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) c_{j,\mu}^{(i)} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) a_{j,\dot{n}_\mu}^{(i)} \quad \text{for } j \in [1, m], \quad i \in [1, s]. \quad (4.22)$$

By (3.5), we have

$$\nu_{P_\infty}(z_r) = r, \quad \text{for } r \geq 0, \quad \text{and} \quad z_{n_u} = w_u \quad \text{with } u = 0, 1, \dots, g. \quad (4.23)$$

Using Lemma 3, (2.2) and (2.8), we obtain that there exists a sequence  $(z_j^\perp)_{j \geq 1}$  such that  $\nu_{P_\infty}(z_j^\perp) = -j$  and

$$\text{Res}_{P_\infty, z}(z_i z_{j+1}^\perp) = S_{-1}(z, z_i z_{j+1}^\perp) = \delta_{i,j} \quad \text{for all } i, j \geq 0. \quad (4.24)$$

We put

$$f_n = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z_{\dot{n}_\mu+1}^\perp. \quad (4.25)$$

Hence

$$\bar{a}_\mu(n) = \text{Res}_{P_\infty, z}(f_n z_{\dot{n}_\mu}) \quad \text{for } 0 \leq \mu \leq m-1, \quad n \in [0, b^m). \quad (4.26)$$

By (2.12) and (4.21), we have  $\delta_{\dot{n}_\mu, n_u} = 0$  for all  $0 \leq u \leq g, \mu \geq 0$ .

Applying (4.23) and (4.24), we derive

$$\text{Res}_{P_\infty, z}(f_n w_u) = \text{Res}_{P_\infty, z} \left( \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z_{\dot{n}_\mu+1}^\perp z_{n_u} \right) \quad (4.27)$$

$$= \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \text{Res}_{P_\infty, z}(z_{\dot{n}_\mu+1}^\perp z_{n_u}) = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \delta_{\dot{n}_\mu, n_u} = 0 \quad \text{for } u = 0, 1, \dots, g, \quad n \geq 0.$$

According to (3.6) and (4.25), we have

$$\begin{aligned} \operatorname{Res}_{P_\infty, z}(f_n k_{i,j}) &= \operatorname{Res}_{P_\infty, z} \left( \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z_{\dot{n}_\mu+1}^\perp \sum_{r=0}^{\infty} a_{j,r}^{(i)} z_r \right) \\ &= \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) a_{j,r}^{(i)} \operatorname{Res}_{P_\infty, z}(z_{\dot{n}_\mu+1}^\perp z_r) = \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) a_{j,r}^{(i)} \delta_{\dot{n}_\mu, r} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) a_{j, \dot{n}_\mu}^{(i)}. \end{aligned}$$

From (4.22), we get

$$\operatorname{Res}_{P_\infty, z}(f_n k_{i,j}) = y_{n,j}^{(i)} \quad \text{for all } j \in [1, m], i \in [1, s], n \in [0, b^m]. \quad (4.28)$$

Using (4.20) and (4.27), we derive

$$\operatorname{Res}_{P_\infty, z} \left( f_{\tilde{n}} \left( \sum_{r=0}^g b_r w_r + \sum_{i=1}^s \sum_{j=1}^{\dot{d}_i e_i} b_{i,j} k_{i,j} \right) \right) = 0 \quad \text{for all } b_i, b_{i,j} \in \mathbb{F}_b.$$

Taking into account that  $(w_0, \dots, w_g, k_{1,1}, \dots, k_{1, \dot{d}_1 e_1}, \dots, k_{s,1}, \dots, k_{s, \dot{d}_s e_s})$  is the basis of  $\mathcal{L}(G + \sum_{i=1}^s \dot{d}_i P_i)$  (see (3.2)), we obtain

$$\operatorname{Res}_{P_\infty, z}(f_{\tilde{n}} \gamma) = 0 \quad \text{for all } \gamma \in \mathcal{L}(\dot{G}) \quad \text{with } \dot{G} = G + \sum_{i=1}^s \dot{d}_i P_i. \quad (4.29)$$

By (4.20), we have

$$\deg(\dot{G} - (m+g+1)P_\infty) = 2g + \sum_{i=1}^s \dot{d}_i e_i - (m+g+1) \geq 2g + m + g - (m+g+1) = 2g - 1.$$

Using the Riemann-Roch theorem, we get

$$\ddot{G} = (\dot{G} - (m+g)P_\infty) \setminus (\dot{G} - (m+g+1)P_\infty) \neq \emptyset.$$

We take  $v \in \ddot{G}$ . Hence  $\nu_{P_\infty}(v) = m+g$ .

From (3.5), we derive  $v = \sum_{r \geq m+g} \hat{b}_r z_r$  with some  $\hat{b}_r \in \mathbb{F}_b$  ( $r \geq m+g$ ) and  $\hat{b}_{m+g} \neq 0$ . According to (4.21), we have  $\dot{n}_{m-1} = m+g$ . Therefore  $v = \sum_{r \geq \dot{n}_{m-1}} \hat{b}_r z_r$ .

Taking into account that  $\tilde{n} \in [b^{m-1}, b^m]$ , we get  $a_{m-1}(\tilde{n}) \neq 0$ .

By (4.24), (4.25) and (4.29), we obtain

$$0 = \operatorname{Res}_{P_\infty, z}(f_{\tilde{n}} v) = \sum_{\mu=0}^{m-1} \sum_{r \geq \dot{n}_{m-1}} a_\mu(\tilde{n}) \hat{b}_r \operatorname{Res}_{P_\infty, z}(z_{\dot{n}_\mu+1}^\perp z_r) = \sum_{\mu=0}^{m-1} \sum_{r \geq \dot{n}_{m-1}} a_\mu(\tilde{n}) \hat{b}_r \delta_{\dot{n}_\mu, r}.$$



Bearing in mind that  $\delta_{\dot{n}_\mu, r} = 1$  for  $\mu \in [0, m-1]$ ,  $r \geq \dot{n}_{m-1}$  if and only if  $\mu = m-1$  and  $r = \dot{n}_{m-1}$  (see (4.21)), we get  $\text{Res}_{P_\infty, z}(f_{\tilde{n}}v) = a_{m-1}(\tilde{n})\hat{b}_{\dot{n}_{m-1}} \neq 0$ . We have a contradiction. Hence Lemma 4 is proved.  $\square$

**Lemma 5.** *Let  $s \geq 2$ ,  $d_i = d_0 e[m\epsilon]$ ,  $1 \leq i \leq s$ ,  $d_{s+1,1} = t + (s-1)d_0 e[m\epsilon]$ ,  $d_{s+1,2} = t-1 + s d_0 e[m\epsilon]$ ,  $d_0 = d+t$ ,  $t = g + e_0 - s$ ,  $e = e_1 \dots e_s$  and  $m \geq 2/\epsilon$ . Then the system  $\{w_0, w_1, \dots, w_g\} \cup \{z^{j+g+1}\}_{d_{s+1,1} \leq j \leq d_{s+1,2}} \cup \{k_{i,j}\}_{1 \leq i \leq s, 1 \leq j \leq d_i}$  of elements of  $F$  is linearly independent over  $\mathbb{F}_b$ .*

**Proof.** Suppose that

$$\alpha := \sum_{j=0}^g b_{0,j} w_j + \sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} k_{i,j} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} z^{j+g+1} = 0$$

for some  $b_{i,j} \in \mathbb{F}_b$  and  $\sum_{j=0}^g |b_{0,j}| + \sum_{i=1}^s \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0$ . Let

$$\beta_1 = \sum_{j=0}^g b_{0,j} w_j, \quad \beta_{2,i} = \sum_{j=1}^{d_i} b_{i,j} k_{i,j}, \quad \beta_2 = \sum_{i=1}^s \beta_{2,i}, \quad \beta_3 = \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} z^{j+g+1}. \quad (4.30)$$

We have

$$\alpha = \beta_1 + \beta_2 + \beta_3 = 0. \quad (4.31)$$

Suppose that  $\sum_{i=1}^s \sum_{j=1}^{d_i} |b_{i,j}| = 0$  and  $\alpha = 0$ . By (4.30) and (4.31), we have  $\beta_1 + \beta_3 = 0$  and  $\nu_{P_\infty}(\beta_1) \geq d_{s+1,1}$ . Taking into account that  $\beta_1 \in \mathcal{L}(G)$  with  $\deg(G) = 2g$ , we obtain from the Riemann-Roch theorem that  $\beta_1 = 0$ . Therefore  $\sum_{j=0}^g |b_{0,j}| = 0$  and  $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| = 0$ . We have a contradiction.

According to [DiPi, Lemma 8.10], we get that if  $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| = 0$  and  $\alpha = 0$ , then  $\sum_{j=0}^g |b_{0,j}| = 0$  and  $\sum_{i=1}^s \sum_{j=1}^{d_i} |b_{i,j}| = 0$ . So, we will consider only the case then  $\sum_{i=1}^s \sum_{j=1}^{d_i} |b_{i,j}| > 0$  and  $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0$ .

Let  $\sum_{j=1}^{d_h} |b_{h,j}| > 0$  for some  $h \in [1, s]$ , and let  $\nu_{P_h}(z) \geq 0$ . By the construction of  $k_{h,j}$ , we have  $\beta_{2,h} \notin \mathcal{L}(G)$  and  $\beta_{2,h} \neq 0$ . Applying (3.3) and (4.30), we obtain  $\nu_P(\beta_{2,h}) \geq -\nu_P(G)$  for any place  $P \neq P_h$  and hence we obtain that  $\nu_{P_h}(\beta_{2,h}) \leq -\nu_{P_h}(G) - 1$  with  $\nu_{P_h}(G) \geq 0$ .

On the other hand, using (3.3) (4.30) and (4.31), we get

$$\nu_{P_h}(\beta_{2,h}) = \nu_{P_h} \left( -\beta_1 - \sum_{i=1, i \neq h}^s \beta_{2,i} - \beta_3 \right)$$

$$\geq \min \left( \nu_{P_h}(\beta_1), \nu_{P_h}(\beta_3), \min_{1 \leq i \leq s, i \neq h} \nu_{P_h}(\beta_{2,i}) \right) \geq -\nu_{P_h}(G).$$

We have a contradiction.

Now let  $\nu_{P_h}(z) \leq -1$ . Bearing in mind that  $\sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0$ , we obtain that  $\beta_3 \neq 0$ , and  $\nu_{P_h}(\beta_3) \leq -d_{s+1,1} - g - 1$ . On the other hand, using (3.3) and (4.31), we have

$$\nu_{P_h}(\beta_3) = \nu_{P_h}(\beta_1 + \beta_2) \geq -\nu_{P_h}(G) - [(d_h - 1)/e_h + 1]e_h \geq -2g - d_h.$$

Taking into account that

$$d_{s+1,1} + g + 1 - (2g + d_h) = t + g + 1 + (s - 2)d_0e[m\epsilon] - 2g \geq t - g + 1 \geq 1,$$

we have a contradiction. Thus Lemma 5 is proved.  $\square$

**Lemma 6.** *Let  $s \geq 2$ ,  $d_0 = d + t$ ,  $t = g + e_0 - s$ ,  $\epsilon = \eta_1(2sd_0e)^{-1}$ ,  $\eta_1 = (1 + \deg((z)_\infty))^{-1}$ ,*

$$\Lambda_1 := \{(y_{n,1}^{(1)}, \dots, y_{n,d_1}^{(1)}, \dots, y_{n,1}^{(s)}, \dots, y_{n,d_s}^{(s)}, \bar{a}_{d_{s+1,1}}(n), \dots, \bar{a}_{d_{s+1,2}}(n)) \mid n \in [0, b^m]\},$$

where

$$d_i = \ddot{m}_i := d_0e[m\epsilon] \quad (1 \leq i \leq s), \quad d_{s+1,1} = \ddot{m}_{s+1} + 1 := t + (s - 1)d_0e[m\epsilon], \quad (4.32)$$

$d_{s+1,2} = \dot{m}_{s+1} := t - 1 + sd_0e[m\epsilon]$ ,  $e = e_1e_2 \cdots e_s$ , and  $n = \sum_{0 \leq j \leq m-1} a_j(n)b^j$ . Then

$$\Lambda_1 = \mathbb{F}_b^{(s+1)d_0e[m\epsilon]}, \quad \text{with } m \geq 9(d+t)es^2\eta_1^{-1}. \quad (4.33)$$

**Proof.** Suppose that (4.33) is not true. Then there exists  $b_{i,j} \in \mathbb{F}_b$  ( $i, j \geq 1$ ) such that

$$\sum_{i=1}^s \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} |b_{s+1,j}| > 0 \quad (4.34)$$

and

$$\sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} y_{n,j}^{(i)} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} \bar{a}_j(n) = 0 \quad \text{for all } n \in [0, b^m]. \quad (4.35)$$

From (4.26) and (4.28), we obtain for  $n \in [0, b^m)$

$$\bar{a}_{j-1}(n) = \text{Res}_{P_\infty, z}(f_n z_{\dot{n}_{j-1}}) \quad \text{and} \quad y_{n,j}^{(i)} = \text{Res}_{P_\infty, z}(f_n k_{i,j}) \quad \text{with} \quad j \in [1, m], \quad i \in [1, s].$$

Applying (3.5) and (4.21), we get  $\dot{n}_{j-1} = g + j$  and  $z_{\dot{n}_{j-1}} = z^{g+j}$  for  $j \geq d_{s+1,1}$ . Hence

$$\sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} \text{Res}_{P_\infty, z}(f_n k_{i,j}) + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} \text{Res}_{P_\infty, z}(f_n z^{g+j+1}) = \text{Res}_{P_\infty, z}(f_n \alpha_1) = 0 \quad (4.36)$$

with

$$\alpha_1 = \sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} k_{i,j} + \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} z^{g+j+1} \quad \text{for} \quad n \in [0, b^m). \quad (4.37)$$

Let

$$b_{0,u} = - \sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} a_{j,n_u}^{(i)}, \quad \beta_1 = \sum_{u=0}^g b_{0,u} w_u, \quad \beta_2 = \sum_{i=1}^s \sum_{j=1}^{d_i} b_{i,j} k_{i,j},$$

$$\beta_3 = \sum_{j=d_{s+1,1}}^{d_{s+1,2}} b_{s+1,j} z^{g+j+1} \quad \text{and} \quad \alpha_2 = \beta_1 + \beta_2 + \beta_3 = \beta_1 + \alpha_1. \quad (4.38)$$

By (4.34) and Lemma 5, we get

$$\alpha_2 \neq 0. \quad (4.39)$$

Consider the local expansion

$$\alpha_2 = \sum_{r=0}^{\infty} \varphi_r z_r \quad \text{with} \quad \varphi_r \in \mathbb{F}_b, \quad r \geq 0. \quad (4.40)$$

Using (3.5), (3.6) and (4.38), we have

$$\varphi_{n_u} = 0 \quad \text{for} \quad 0 \leq u \leq g. \quad (4.41)$$

From (4.27), we derive  $\text{Res}_{P_\infty, z}(f_n w_u) = 0$  ( $0 \leq u \leq g$ ). By (4.36) and (4.38), we get

$$\text{Res}_{P_\infty, z}(f_n \beta_1) = 0 \quad \text{and} \quad \text{Res}_{P_\infty, z}(f_n \alpha_2) = 0 \quad \text{for all} \quad n \in [0, b^m).$$

Applying (4.24), (4.25) and (4.40), we obtain

$$\begin{aligned} \text{Res}_{P_\infty, z}(f_n \alpha_2) &= \text{Res}_{P_\infty, z} \left( \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) z_{\dot{n}_\mu+1}^\perp \sum_{r=0}^{\infty} \varphi_r z_r \right) \\ &= \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) \varphi_r \text{Res}_{P_\infty, z}(z_{\dot{n}_\mu+1}^\perp z_r) = \sum_{\mu=0}^{m-1} \sum_{r=0}^{\infty} \bar{a}_\mu(n) \varphi_r \delta_{\dot{n}_\mu, r} = \sum_{\mu=0}^{m-1} \bar{a}_\mu(n) \varphi_{\dot{n}_\mu} = 0 \end{aligned}$$

for all  $n \in [0, b^m)$ .

Hence  $\varphi_{\dot{n}_\mu} = 0$  for  $\mu \in [0, m-1]$ . According to (4.21) and (4.41), we have

$$\varphi_r = 0 \quad \text{for } r \in [0, m+g]. \quad (4.42)$$

Therefore

$$\nu_{P_\infty}(\alpha_2) > m+g. \quad (4.43)$$

From (3.3) and (4.38), we derive

$$\beta_1 + \beta_2 \in \mathcal{L}\left(G + \sum_{i=1}^s [(d_i - 1)/e_i + 1] P_i\right) \quad \text{and} \quad \beta_3 \in \mathcal{L}((d_{s+1,2} + g + 1)(z)_\infty).$$

By (4.43), we obtain

$$\alpha_2 \in \mathcal{L}(G_1) \quad \text{with} \quad G_1 = G + \sum_{i=1}^s [(d_i - 1)/e_i + 1] P_i + (d_{s+1,2} + g + 1)(z)_\infty - (m + g + 1)P_\infty.$$

Using (4.32), we have

$$\begin{aligned} \deg(G_1) &= 2g + \sum_{i=1}^s d_i + (d_{s+1,2} + g + 1)\deg((z)_\infty) - (m + g + 1) \\ &= 2g + sd_0e[m\epsilon] + (t + g + sd_0e[m\epsilon])(\eta_1^{-1} - 1) - (m + g + 1) \\ &\leq 2g + (t + g)(\eta_1^{-1} - 1) + sd_0em\epsilon\eta_1^{-1} - (m + g + 1) \\ &= g - 1 + (t + g)(\eta_1^{-1} - 1) - m(1 - sd_0e\epsilon\eta_1^{-1}) = g - 1 + (t + g)(\eta_1^{-1} - 1) - m/2 < 0 \end{aligned}$$

for  $m \geq 9(d+t)es^2\eta_1^{-1} > 2(g-1) + 2(t+g)(\eta_1^{-1} - 1)$  and  $d = g + e_0$ . Hence  $\alpha_2 = 0$ . By (4.39), we have a contradiction. Therefore assertion (4.35) is not true. Thus Lemma 6 is proved.  $\square$

**End of the proof of Theorem 2.** Using Lemma 4 and Theorem I, we get that  $(\mathbf{x}(n))_{n \geq 0}$  is a  $d$ -admissible digital  $(t, s)$  sequence with  $d = g + e_0$  and  $t = g + e_0 - s$ . Applying Lemma 6 and Corollary 3 with  $B'_i = \emptyset$ ,  $1 \leq i \leq s + 1$ ,  $B = 0$  and  $\hat{e} = e = e_1 e_2 \cdots e_s$ , we get the first assertion in Theorem 2.

Consider the second assertion in Theorem 2 :  
Let, for example,  $i_0 = s$ , i.e.

$$\nu_{P_\infty}(k_{s,j}) \geq \eta_2 j \quad \text{for } j \geq m/2 - t, \quad \text{and } \eta_2 \in (0, 1). \quad (4.44)$$

From (1.4), Lemma 4 and Theorem I, we get that  $(\mathbf{x}(n))_{0 \leq n < b^m}$  is a  $d$ -admissible digital  $(t, m, s)$  net with  $d = g + e_0$  and  $t = g + e_0 - s$ .

We apply Corollary 2 with  $\dot{s} = s \geq 3$ ,  $B_i = \emptyset$ ,  $1 \leq i \leq s$ ,  $B = 0$ ,  $\tilde{r} = 0$ ,  $m = \tilde{m}$ ,  $\hat{e} = e = e_1 e_2 \cdots e_s$ ,  $d_0 = d + t$ ,  $t = g + e_0 - s$  and  $e_0 = e_1 + \dots + e_s$ . In order to prove the second assertion in Theorem 2, it is sufficient to verify that

$$\Lambda_2 = \mathbb{F}_b^{sd_0 e[m\epsilon]} \quad \text{for } m \geq 8(d+t)e(s-1)^2 \eta_2^{-1} + 2(1+2g+\eta_2 t) \eta_2^{-1} (1-\eta_2)^{-1}, \quad (4.45)$$

where

$$\Lambda_2 = \{(y_{n,1}^{(1)}, \dots, y_{n,d_1}^{(1)}, \dots, y_{n,1}^{(s-1)}, \dots, y_{n,d_{s-1}}^{(s-1)}, y_{n,d_{s,1}}^{(s)}, \dots, y_{n,d_{s,2}}^{(s)}) \mid n \in [0, b^m)\}$$

with

$$d_i = \dot{m}_i := d_0 e[m\epsilon], \quad i \in [1, s), \quad d_{s,1} = \ddot{m}_s + 1 := m - t + 1 - (s-1)d_0 e[m\epsilon], \quad (4.46)$$

$$d_{s,2} = \dot{m}_s := m - t - (s-2)d_0 e[m\epsilon], \quad \text{and } \epsilon = \eta_2(2(s-1)d_0 e)^{-1}.$$

Suppose that (4.45) is not true. Then there exists  $b_{i,j} \in \mathbb{F}_b$  ( $i, j \geq 1$ ) such that

$$\sum_{i=1}^{s-1} \sum_{j=1}^{d_i} |b_{i,j}| + \sum_{j=d_{s,1}}^{d_{s,2}} |b_{s,j}| > 0 \quad (4.47)$$

and

$$\sum_{i=1}^{s-1} \sum_{j=1}^{d_i} b_{i,j} y_{n,j}^{(i)} + \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} y_{n,j}^{(s)} = 0 \quad \text{for all } n \in [0, b^m).$$

Similarly to (4.36), we get

$$\text{Res}_{P_\infty, z}(f_n \alpha_1) = 0 \quad \text{for all } n \in [0, b^m), \quad \text{with } \alpha_1 = \alpha_2 - \beta_1$$

where  $\alpha_2 = \beta_1 + \beta_2 + \beta_3$ , with

$$\beta_1 = \sum_{u=0}^g b_{0,u} w_u, \quad \beta_2 = \sum_{i=1}^{s-1} \sum_{j=1}^{d_i} b_{i,j} k_{i,j} \quad \text{and} \quad \beta_3 = \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} k_{s,j} \quad (4.48)$$

and  $b_{0,u} = -\sum_{i=1}^{s-1} \sum_{j=1}^{d_i} b_{i,j} a_{j,n_u}^{(i)} - \sum_{j=d_{s,1}}^{d_{s,2}} b_{s,j} a_{j,n_u}^{(s)}$ . Consider the local expansions

$$\beta_1 + \beta_2 = \sum_{r=0}^{\infty} \dot{\varphi}_r z_r \quad \text{and} \quad \beta_3 = \sum_{r=0}^{\infty} \ddot{\varphi}_r z_r \quad \text{with} \quad \varphi_{i,r} \in \mathbb{F}_b \quad i = 1, 2, \quad r \geq 0.$$

Analogously to (4.42), we obtain

$$\dot{\varphi}_r + \ddot{\varphi}_r = 0 \quad \text{for} \quad r \in [0, m+g]. \quad (4.49)$$

Using (4.44), (4.46) and (4.48), we get

$$\nu_{P_\infty}(k_{s,j}) \geq \eta_2 j \quad \text{for} \quad j \geq d_{s,1} \geq m/2 - t, \quad \text{and} \quad \ddot{\varphi}_r = 0 \quad \text{for} \quad r \leq [\eta_2 d_{s,1}] - 1.$$

Therefore  $\dot{\varphi}_r = 0$  for  $r \leq [\eta_2 d_{s,1}] - 1$ . Hence

$$\nu_{P_\infty}(\beta_1 + \beta_2) \geq [\eta_2 d_{s,1}].$$

By (4.48), we obtain

$$\beta_1 + \beta_2 \in \mathcal{L}(G_2) \quad \text{with} \quad G_2 = G + \sum_{i=1}^{s-1} [(d_i - 1)/e_i + 1] P_i - [\eta_2 d_{s,1}] P_\infty.$$

According to (4.45) and (4.46), we have

$$\deg(G_2) = 2g + \sum_{i=1}^{s-1} d_i - [\eta_2 d_{s,1}] = 2g + (s-1)d_0 e[m\epsilon] - [\eta_2(m-t+1 - (s-1)d_0 e[m\epsilon])]$$

$$\leq 2g + (s-1)d_0 e[m\epsilon] - \eta_2(m-t+1 - (s-1)d_0 e[m\epsilon]) + 1 = (1 + \eta_2)(s-1)d_0 e[m\epsilon] - m\eta_2 + 2g + 1 + \eta_2(t-1) \leq m\eta_2((1 + \eta_2)/2 - 1) + 1 + 2g + \eta_2 t < 0$$

for  $m > 2(1 + 2g + \eta_2 t)\eta_2^{-1}(1 - \eta_2)^{-1}$ . Hence  $\beta_1 + \beta_2 = 0$ .

By [DiPi, Lemma 8.10] (or Lemma 5), we get that  $b_{i,j} = 0$  for all  $j \in [1, d_i]$ ,  $i \in [1, s-1]$  and  $b_{0,j} = 0$  for  $j \in [0, g]$ .

From (4.49) we have  $\ddot{\varphi}_r = 0$  for  $r \in [0, m + g]$ . Thus  $\nu_{P_\infty}(\beta_3) \geq m + g + 1$ . Applying (4.48), we derive

$$\beta_3 \in \mathcal{L}(G_3) \quad \text{with} \quad G_3 = G + [(d_{s,2} - 1)/e_s + 1]P_s - (m + g + 1)P_\infty.$$

By (4.46), we obtain

$$\deg(G_3) = 2g + m - t - (s - 2)d_0e[m\epsilon] + e_s - m - g - 1 \leq g - t - 1 + e_s - (s - 2)d_0e[m\epsilon] < 0$$

for  $m \geq \epsilon^{-1}$  and  $s \geq 3$ . Hence  $\beta_3 = 0$ . Using (3.2) and (4.48), we get that  $b_{s,j} = 0$  for all  $j \in [d_{s,1}, d_{s,2}]$ .

By (4.47), we have a contradiction. Thus assertions (4.45) and (3.9) are true. Therefore Theorem 2 is proved.  $\square$

### 4.3 Niederreiter-Özbudak nets. Proof of Theorem 3

Let

$$m = m_i e_i + r_i, \quad \text{with} \quad 0 \leq r_i < e_i, \quad 1 \leq i \leq s \quad \text{and} \quad \tilde{r}_0 = \sum_{i=1}^{s-1} r_i, \quad r_0 = \sum_{i=1}^s r_i. \quad (4.50)$$

**Lemma 7.** *There exists a divisor  $\tilde{G}$  of  $F/\mathbb{F}_b$  with  $\deg(\tilde{G}) = g - 1 + \tilde{r}_0$ , such that  $\nu_{P_i}(\tilde{G}) = 0$  for  $1 \leq i \leq s$ , and*

$$\mathcal{N}_m(P_1, \dots, P_s; G) = \mathcal{N}_m(P_1, \dots, P_s; \hat{G}), \quad \text{where} \quad \hat{G} = m_1 P_1 + \dots + m_{s-1} P_{s-1} + \tilde{G}.$$

**Proof.** We have  $\nu_{P_i}(G) = a_i$  and  $\nu_{P_i}(t_i) = 1$  for  $1 \leq i \leq s$ . Using the Approximation Theorem, we obtain that there exists  $y \in F$ , such that

$$\nu_{P_i}(y - t_i^{a_i - m_i}) = a_i + 1, \quad \text{for} \quad 1 \leq i \leq s - 1, \quad \nu_{P_s}(y - t_s^{a_s}) = a_s + m_s + 1. \quad (4.51)$$

Let  $\dot{f} = fy$  and  $\hat{G} = G - \text{div}(y)$ . We note

$$f \in \mathcal{L}(G) \Leftrightarrow \text{div}(f) + G \geq 0 \Leftrightarrow \text{div}(fy) + G - \text{div}(y) \geq 0 \Leftrightarrow \dot{f} = fy \in \mathcal{L}(\hat{G}). \quad (4.52)$$

It is easy to see that  $\nu_{P_i}(\hat{G}) = m_i$  ( $1 \leq i \leq s - 1$ ),  $\nu_{P_s}(\hat{G}) = 0$  and  $\deg(\hat{G}) = \deg(G) = m(s - 1) + g - 1$ . Let  $\tilde{G} = \hat{G} - m_1 P_1 - \dots - m_{s-1} P_{s-1}$ . We get  $\nu_{P_i}(\tilde{G}) = 0$  for  $1 \leq i \leq s$ . Hence

$$\deg(\tilde{G}) = m(s - 1) + g - 1 - e_1 m_1 - \dots - e_{s-1} m_{s-1} = g - 1 + \tilde{r}_0.$$

Let  $\dot{f}_{i,j} = S_j(t_i, \dot{f})$  (see (3.10)). By (4.51), we have

$$\dot{f}_{i,-j} = f_{i,-a_i+m_i-j} \quad 1 \leq i \leq s-1, \quad \text{and} \quad \dot{f}_{s,m_s-j} = f_{s,-a_s+m_s-j} \quad \text{with} \quad 1 \leq j \leq m_s.$$

Using notations (3.11), we get

$$\theta_i^{(\hat{G})}(\dot{f}) = (\mathbf{0}_{r_i}, \vartheta_i(\dot{f}_{i,-1}), \dots, \vartheta_i(\dot{f}_{i,-m_i})) = (\mathbf{0}_{r_i}, \vartheta_i(f_{i,-a_i+m_i-1}), \dots, \vartheta_i(f_{i,-a_i})) = \theta_i^{(G)}(f)$$

for  $1 \leq i \leq s-1$ , and

$$\theta_s^{(\hat{G})}(\dot{f}) = (\mathbf{0}_{r_s}, \vartheta_s(\dot{f}_{s,m_s-1}), \dots, \vartheta_s(\dot{f}_{s,0})) = (\mathbf{0}_{r_s}, \vartheta_s(f_{s,-a_s+m_s-1}), \dots, \vartheta_s(f_{s,-a_s})) = \theta_s^{(G)}(f).$$

By (3.12), we have

$$\theta^{(\hat{G})}(\dot{f}) := (\theta_1^{(\hat{G})}(\dot{f}), \dots, \theta_s^{(\hat{G})}(\dot{f})) = (\theta_1^{(G)}(f), \dots, \theta_s^{(G)}(f)) = \theta^{(G)}(f)$$

for all  $f \in \mathcal{L}(G)$ . From (3.13) and (4.52), we obtain the assertion of Lemma 7.  $\square$

By Lemma 7, we can take  $\hat{G}$  instead of  $G$ . Hence

$$G = m_1 P_1 + \dots + m_{s-1} P_{s-1} + \tilde{G}, \quad \text{and} \quad a_i = m_i, \quad 1 \leq i \leq s-1, \quad a_s = 0. \quad (4.53)$$

Let  $\vartheta_i = (\vartheta_{i,1}, \dots, \vartheta_{i,e_i})$ . From (3.11), we get for  $0 \leq \check{j}_i \leq m_i - 1$ ,  $1 \leq \hat{j}_i \leq e_i$ , that

$$\theta_i^{(G)}(f) = (\theta_{i,1}(f), \dots, \theta_{i,m}(f)) = (\mathbf{0}_{r_i}, \vartheta_i(f_{i,-1}), \dots, \vartheta_i(f_{i,-m_i})), \quad 1 \leq i \leq s-1,$$

with  $\theta_{i,r_i+\check{j}_i e_i + \hat{j}_i}(f) = \vartheta_{i,\hat{j}_i}(f_{i,-\check{j}_i-1})$ , and

$$\theta_s^{(G)}(f) = (\theta_{s,1}(f), \dots, \theta_{s,m}(f)) = (\mathbf{0}_{r_s}, \vartheta_s(f_{s,m_s-1}), \dots, \vartheta_s(f_{s,0})), \quad (4.54)$$

with  $\theta_{s,r_s+\check{j}_s e_s + \hat{j}_s}(f) = \vartheta_{s,\hat{j}_s}(f_{s,m_s-\check{j}_s-1})$ .

**Lemma 8.** *Let  $\vartheta_i = (\vartheta_{i,1}, \dots, \vartheta_{i,e_i}) : F_{P_i} \rightarrow \mathbb{F}_b^{e_i}$  be an  $\mathbb{F}_b$ -linear vector space isomorphism. Then there exists an  $\mathbb{F}_b$ -linear vector space isomorphism  $\vartheta_i^\perp = (\vartheta_{i,1}^\perp, \dots, \vartheta_{i,e_i}^\perp) : F_{P_i} \rightarrow \mathbb{F}_b^{e_i}$  such that*

$$\text{Tr}_{F_{P_i}/\mathbb{F}_b}(\dot{x}\ddot{x}) = \sum_{j=1}^{e_i} \vartheta_{i,j}(\dot{x}) \vartheta_{i,j}^\perp(\ddot{x}) \quad \text{for all} \quad \dot{x}, \ddot{x} \in F_{P_i}, \quad 1 \leq i \leq s.$$

**Proof.** Using Theorem F, we get that there exists  $\beta_{i,j} \in F_{P_i}$  such that

$$\vartheta_{i,j}(y) = \text{Tr}_{F_{P_i}/\mathbb{F}_b}(y \beta_{i,j}) \quad \text{for} \quad 1 \leq j \leq e_i, \quad (4.55)$$



and  $(\beta_{i,1}, \dots, \beta_{i,e_i})$  is the basis of  $F_{P_i}$  over  $\mathbb{F}_b$  ( $1 \leq i \leq s$ ). Applying Theorem G, we obtain that there exists a basis  $(\beta_{i,1}^\perp, \dots, \beta_{i,e_i}^\perp)$  of  $F_{P_i}$  over  $\mathbb{F}_b$  such that

$$\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{i,j_1}\beta_{i,j_2}^\perp) = \delta_{j_1,j_2} \quad \text{with} \quad 1 \leq j_1, j_2 \leq e_i.$$

Let  $\dot{x} = \sum_{j=1}^{e_i} \dot{\gamma}_j \beta_{i,j}^\perp$ ,  $\ddot{x} = \sum_{j=1}^{e_i} \ddot{\gamma}_j \beta_{i,j}$  and let

$$\vartheta_{i,j}^\perp(\ddot{x}) := \ddot{\gamma}_j = \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\ddot{x} \beta_{i,j}^\perp). \quad (4.56)$$

By (4.55), we have  $\dot{\gamma}_j = \vartheta_{i,j}(\dot{x})$ . Now, we get

$$\mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\dot{x}\ddot{x}) = \sum_{j_1,j_2=1}^{e_i} \dot{\gamma}_{j_1} \ddot{\gamma}_{j_2} \mathrm{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{i,j_1}^\perp \beta_{i,j_2}) = \sum_{j=1}^{e_i} \dot{\gamma}_j \ddot{\gamma}_j = \sum_{j=1}^{e_i} \vartheta_{i,j}(\dot{x}) \vartheta_{i,j}^\perp(\ddot{x}).$$

Hence Lemma 8 is proved.  $\square$

We consider the  $H$ -differential  $dt_s$ . Let  $\omega$  be the corresponding Weil differential,  $\mathrm{div}(\omega)$  the divisor of  $\omega$ , and  $W := \mathrm{div}(dt_s) = \mathrm{div}(\omega)$ . By (2.4) and (2.6), we have

$$\deg(W) = 2g - 2 \quad \text{and} \quad \nu_{P_s}(W) = \nu_{P_s}(dt_s) = \nu_{P_s}(dt_s/dt_s) = 0. \quad (4.57)$$

Using notations of Lemma 7, we define

$$G^\perp = m_s P_s - \tilde{G} + W, \quad \text{where} \quad \deg(\tilde{G}) = g - 1 + \tilde{r}_0 \quad \text{and} \quad \nu_{P_i}(\tilde{G}) = 0 \quad (4.58)$$

for  $1 \leq i \leq s$ . Let  $a_i^\perp := \nu_{P_i}(G^\perp - W)$  for  $1 \leq i \leq s$ . We obtain from (4.58) that  $a_i^\perp = 0$  for  $1 \leq i \leq s-1$  and  $a_s^\perp = m_s$ . Let  $f^\perp \in \mathcal{L}(G^\perp)$ , then  $\mathrm{div}(f^\perp) + W + G^\perp - W \geq 0$  and  $\nu_{P_i}(\mathrm{div}(f^\perp) + W) \geq -\nu_{P_i}(G^\perp - W)$ . Applying (2.6), we get

$$\nu_{P_i}(f^\perp dt_s) = \nu_{P_i}(f^\perp) + \nu_{P_i}(W) \geq -\nu_{P_i}(G^\perp - W) = -a_i^\perp, \quad \text{with} \quad a_i^\perp = 0, \quad (4.59)$$

$1 \leq i \leq s-1$ , and  $a_s^\perp = m_s$  for  $f^\perp \in \mathcal{L}(G^\perp)$ . According to Proposition A, we have that there exists  $\tau_i \in F$ , such that

$$dt_s = \tau_i dt_i, \quad 1 \leq i \leq s. \quad (4.60)$$

From (2.4) and (4.59), we get

$$\nu_{P_i}(f^\perp \tau_i) = \nu_{P_i}(f^\perp \tau_i dt_i) = \nu_{P_i}(f^\perp dt_s) \geq -a_i^\perp, \quad 1 \leq i \leq s.$$

By (2.2), we have the local expansions

$$f^\perp \tau_i := \sum_{j=-a_i^\perp}^{\infty} S_j(t_i, f^\perp \tau_i) t_i^j, \quad \text{where all } S_j(t_i, f^\perp \tau_i) \in F_{P_i} \quad (4.61)$$

for  $1 \leq i \leq s$  and  $f^\perp \in \mathcal{L}(G^\perp)$ . We denote  $S_j(t_i, f^\perp \tau_i)$  by  $f_{i,j}^\perp$ .

Using (2.7), (2.8) and (4.56), we denote

$$\vartheta_{i,\hat{j}_i}^\perp(f_{i,\check{j}_i}^\perp) := \text{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{i,\hat{j}_i}^\perp f_{i,\check{j}_i}^\perp) = \text{Res}_{P_i,t_i}(\beta_{i,\hat{j}_i}^\perp t_i^{-\check{j}_i-1} f^\perp \tau_i) \quad (4.62)$$

and  $\vartheta_i^\perp = (\vartheta_{i,1}^\perp, \dots, \vartheta_{i,e_i}^\perp)$  with  $1 \leq \hat{j}_i \leq e_i$ ,  $-a_i^\perp \leq \check{j}_i \leq -a_i^\perp + m_i - 1$ ,  $1 \leq i \leq s$ .

For  $f^\perp \in \mathcal{L}(G^\perp)$ , the image of  $f^\perp$  under  $\dot{\theta}_i^\perp$ , for  $1 \leq i \leq s$ , is defined as

$$\dot{\theta}_i^\perp(f^\perp) = (\dot{\theta}_{i,1}^\perp(f^\perp), \dots, \dot{\theta}_{i,m}^\perp(f^\perp)) := (\vartheta_i^\perp(f_{i,-a_i^\perp}^\perp), \dots, \vartheta_i^\perp(f_{i,-a_i^\perp+m_i-1}^\perp), \mathbf{0}_{r_i}) \in \mathbb{F}_b^m,$$

It is easy to verify that

$$\dot{\theta}_{i,\hat{j}_i e_i + \hat{j}_i}^\perp(f^\perp) = \vartheta_{i,\hat{j}_i}^\perp(f_{i,\check{j}_i}^\perp), \quad \text{for } 1 \leq \hat{j}_i \leq e_i, \quad 0 \leq \check{j}_i \leq m_i - 1, \quad (4.63)$$

$$1 \leq i \leq s-1 \quad \text{and} \quad \dot{\theta}_{s,\check{j}_s e_s + \hat{j}_s}^\perp(f^\perp) = \vartheta_{s,\hat{j}_s}^\perp(f_{s,-m_s+\check{j}_s}^\perp), \quad 0 \leq \check{j}_s \leq m_s - 1. \quad (4.64)$$

Let

$$\dot{\theta}^{(G,\perp)}(f^\perp) := (\dot{\theta}_1^\perp(f^\perp), \dots, \dot{\theta}_s^\perp(f^\perp)) \in \mathbb{F}_b^{ms}. \quad (4.65)$$

Let  $\varphi_i = (\varphi_{i,1}, \dots, \varphi_{i,r_i})$  with  $\varphi_{i,j} \in \mathbb{F}_b$  ( $1 \leq j \leq r_i$ ,  $1 \leq i \leq s$ ), and let

$$\Phi = \{\varphi = (\varphi_1, \dots, \varphi_s) \mid \varphi_i \in \mathbb{F}_b^{r_i}, \quad i = 1, \dots, s\} \quad \text{with} \quad \dim(\Phi) = r_0 = \sum_{i=1}^s r_i. \quad (4.66)$$

Now, we set

$$\theta^{(G,\perp)}(f^\perp, \varphi) := (\theta_1^\perp(f^\perp, \varphi), \dots, \theta_s^\perp(f^\perp, \varphi)) \in \mathbb{F}_b^{ms}, \quad (4.67)$$

where

$$\theta_i^\perp(f^\perp, \varphi) = (\theta_{i,1}^\perp(f^\perp, \varphi), \dots, \theta_{i,m}^\perp(f^\perp, \varphi)) := (\varphi_i, \dot{\theta}_{i,1}^\perp(f^\perp), \dots, \dot{\theta}_{i,m-r_i}^\perp(f^\perp)) \in \mathbb{F}_b^m.$$

We define the  $\mathbb{F}_b$ -linear maps

$$\theta^{(G,\perp)} : (\mathcal{L}(G^\perp), \Phi) \rightarrow \mathbb{F}_b^{ms}, \quad (f^\perp, \varphi) \mapsto \theta^{(G,\perp)}(f^\perp, \varphi) \quad (4.68)$$

$$\text{and } \dot{\theta}^{(G,\perp)} : \mathcal{L}(G^\perp) \rightarrow \mathbb{F}_b^{ms}, \quad f^\perp \mapsto \dot{\theta}^{(G,\perp)}(f^\perp).$$

The images of  $\theta^{(G,\perp)}$  and  $\dot{\theta}^{(G,\perp)}$  are denoted by

$$\begin{aligned} \Xi_m &:= \{\theta^{(G,\perp)}(f^\perp, \varphi) \mid f^\perp \in \mathcal{L}(G^\perp), \varphi \in \Phi\} \\ \text{and } \dot{\Xi}_m &:= \{\dot{\theta}^{(G,\perp)}(f^\perp) \mid f^\perp \in \mathcal{L}(G^\perp)\}. \end{aligned} \quad (4.69)$$

**Lemma 9** *With notation as above, we have  $\ker(\theta^{(G,\perp)}) = \mathbf{0}$  and*

$$\delta_m^\perp(\dot{\Xi}_m) \leq m + g - 1 + e_0 - r_0.$$

**Proof.** Consider (4.57)-(4.60). Let  $f^\perp \in \mathcal{L}(G^\perp) \setminus \{0\}$ , and let

$$\nu_{P_i}(f^\perp \tau_i) = d_i \quad \text{for } 1 \leq i \leq s-1, \quad \nu_{P_s}(f^\perp) = d_s - m_s. \quad (4.70)$$

We see that

$$\operatorname{div}(f^\perp) + G^\perp \geq 0, \quad \text{with } G^\perp = m_s P_s - \tilde{G} + W \quad \text{and } W = (dt_s). \quad (4.71)$$

Hence

$$\nu_P(\operatorname{div}(f^\perp) + m_s P_s - \tilde{G} + W) \geq 0, \quad \text{for all } P \in \mathbb{P}_F. \quad (4.72)$$

By (2.4) and (2.6), we obtain  $\nu_{P_i}(W) = \nu_{P_i}(dt_s) = \nu_{P_i}(\tau_i)$ ,  $1 \leq i \leq s$ . Bearing in mind (4.70) and that  $\nu_{P_i}(\tilde{G}) = 0$  for  $i \in [1, s]$ , we get

$$\nu_{P_i}(\operatorname{div}(f^\perp) + m_s P_s - \tilde{G} + W) = d_i \geq 0, \quad 1 \leq i \leq s.$$

Therefore

$$\nu_{P_i}(\operatorname{div}(f^\perp) + \dot{G}) \geq 0 \quad \text{for } f^\perp \in \mathcal{L}(G^\perp) \setminus \{0\}, \quad \text{where } \dot{G} = G^\perp - \sum_{i=1}^s d_i P_i$$

and  $G^\perp = m_s P_s - \tilde{G} + W$ . Taking into account that  $f^\perp \in \mathcal{L}(G^\perp) \setminus \{0\}$ , we obtain

$$0 \leq \deg(\dot{G}) = \deg\left(G^\perp - \sum_{i=1}^s d_i P_i\right) = \deg(G^\perp) - \sum_{i=1}^s d_i e_i.$$

By (4.57), (4.58) and (4.50), we get

$$\sum_{i=1}^s d_i e_i \leq \deg(m_s P_s - \tilde{G} + W) = m_s e_s - (g - 1 + \tilde{r}_0) + 2g - 2 = m - r_0 + g - 1.$$

According to (4.61), (4.62) and (4.70), we obtain

$$f_{i,a_i^\perp+j}^\perp = 0 \quad \text{for } 0 \leq j < d_i \quad \text{and} \quad f_{i,a_i^\perp+d_i}^\perp \neq 0, \quad 1 \leq i \leq s.$$

From (2.22), (4.64) and Lemma 8, we have

$$v_m^\perp(\dot{\theta}_i^\perp(f^\perp)) \leq (d_i + 1)e_i \quad \text{for } 1 \leq i \leq s.$$

Applying (4.65) and (2.23), we derive

$$V_m^\perp(\dot{\theta}^{(G,\perp)}(f^\perp)) \leq \sum_{i=1}^s (d_i + 1)e_i \leq m + g - 1 + e_0 - r_0.$$

By (2.24),  $\delta_m^\perp(\Xi_m) \leq m + g - 1 + e_0 - r_0$ . Taking into account (2.22) and that  $s \geq 3$ , we get  $\ker(\theta^{(G,\perp)}) = \mathbf{0}$ .

Therefore Lemma 9 is proved.  $\square$

**Lemma 10.** *With notation as above, we have that  $\dim(\Xi_m) = m$ .*

**Proof.** By (4.57) and (4.58), we have

$$\deg(G^\perp) = \deg(m_s P_s - \tilde{G} + W) = m_s e_s - \deg(\tilde{G}) + 2g - 2 = m - r_s + 2g - 2 - \tilde{r}_0 - g + 1.$$

Using (4.50) and the Riemann-Roch theorem, we obtain for  $m \geq g + e_0 - 1 \geq g + r_0$  that

$$\dim(\mathcal{L}(G^\perp)) = \deg(m_s P_s - \tilde{G} + W) - g + 1 = m - r_0 + 2g - 2 - 2g + 2 = m - r_0.$$

From (4.66), we have  $\dim(\Phi) = r_0$ . Hence

$$\dim((\mathcal{L}(G^\perp), \Phi)) = \dim(\mathcal{L}(G^\perp)) + \dim(\Phi) = m - r_0 + r_0 = m.$$

By Lemma 9, we get  $\ker(\theta^{(G,\perp)}) = \mathbf{0}$ . Bearing in mind that  $\theta^{(G,\perp)}((\mathcal{L}(G^\perp), \Phi)) = \Xi_m$ , we obtain the assertion of Lemma 10.  $\square$

**Lemma 11.** *Let  $f \in \mathcal{L}(G)$ , and  $f^\perp \in \mathcal{L}(G^\perp)$ . Then*

$$\sum_{i=1}^s \text{Res}_{P_i}(f f^\perp dt_s) = 0, \tag{4.73}$$

$$\operatorname{Res}_{P_i}(ff^\perp dt_s) = \sum_{j=0}^{m_i-1} \operatorname{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{i,-j-1} f_{i,j}^\perp), \quad 1 \leq i \leq s-1 \quad (4.74)$$

$$\text{and } \operatorname{Res}_{P_s}(ff^\perp dt_s) = \sum_{j=0}^{m_s-1} \operatorname{Tr}_{F_{P_s}/\mathbb{F}_b}(f_{s,m_s-j-1} f_{s,-m_s+j}^\perp). \quad (4.75)$$

**Proof.** By (4.53) and (4.58), we have

$$G = m_1 P_1 + \dots + m_{s-1} P_{s-1} + \tilde{G}, \quad \text{and} \quad G^\perp = m_s P_s - \tilde{G} + W.$$

Bearing in mind that  $\operatorname{div}(f) + G \geq 0$ ,  $\operatorname{div}(f^\perp) + G^\perp \geq 0$  and that  $W = \operatorname{div}(dt_s)$ , we obtain

$$\operatorname{div}(f) + \sum_{i=1}^s m_i P_i + \tilde{G} + \operatorname{div}(f^\perp) - \tilde{G} + W = \operatorname{div}(f) + \operatorname{div}(f^\perp) + \sum_{i=1}^s m_i P_i + \operatorname{div}(dt_s) \geq 0.$$

From (2.6), we derive

$$\nu_P(ff^\perp dt_s) = \nu_P(ff^\perp) + \nu_P(\operatorname{div}(dt_s)) \geq 0 \quad \text{and} \quad \operatorname{Res}_P(ff^\perp dt_s) = 0$$

for all  $P \in \mathbb{P}_f \setminus \{P_1, \dots, P_s\}$ .

Applying the Residue Theorem, we get assertion (4.73).

By (3.10) and (4.61), we derive

$$\begin{aligned} \operatorname{Res}_{P_s}(ff^\perp dt_s) &= \operatorname{Res}_{P_s} \left( \sum_{j_1=0}^{\infty} S_{j_1}(t_s, f) t_s^{j_1} \sum_{j_2=-m_s}^{\infty} S_{j_2}(t_s, f^\perp) t_s^{j_2} dt_s \right) \\ &= \sum_{j_1=0}^{\infty} \sum_{j_2=-m_s}^{\infty} \operatorname{Res}_{P_s}(S_{j_1}(t_s, f) S_{j_2}(t_s, f^\perp) t_s^{j_1+j_2} dt_s) \\ &= \sum_{0 \leq j_1 \leq m_s-1, j_1+j_2=-1} \operatorname{Tr}_{F_{P_s}/\mathbb{F}_b}(S_{j_1}(t_s, f) S_{j_2}(t_s, f^\perp)) \\ &= \sum_{j=0}^{m_s-1} \operatorname{Tr}_{F_{P_s}/\mathbb{F}_b}(S_{m_s-j-1}(t_s, f) S_{-m_s+j}(t_s, f^\perp)) = \sum_{j=0}^{m_s-1} \operatorname{Tr}_{F_{P_s}/\mathbb{F}_b}(f_{s,m_s-j-1} f_{s,-m_s+j}^\perp). \end{aligned}$$

Hence assertion (4.75) is proved.

Analogously, using (4.60), we have

$$\operatorname{Res}_{P_i}(ff^\perp dt_s) = \operatorname{Res}_{P_i}(ff^\perp \tau_i dt_i) = \operatorname{Res}_{P_i} \left( \sum_{j_1=-m_i}^{\infty} S_{j_1}(t_i, f) t_i^{j_1} \sum_{j_2=0}^{\infty} S_{j_2}(t_i, f^\perp \tau_i) t_i^{j_2} dt_i \right)$$

$$\begin{aligned}
&= \sum_{0 \leq j_2 \leq m_i - 1, j_1 + j_2 = -1} \text{Tr}_{F_{P_i}/\mathbb{F}_b} (S_{j_1}(t_i, f) S_{j_2}(t_i, f^\perp \tau_i)), \\
&= \sum_{j=0}^{m_i-1} \text{Tr}_{F_{P_i}/\mathbb{F}_b} (f_{i,-j-1} f_{i,j}^\perp), \quad \text{for } 1 \leq i \leq s-1.
\end{aligned}$$

Thus Lemma 11 is proved.  $\square$

**Lemma 12.** *With notation as above, we have  $\Xi_m = \mathcal{N}^\perp(P_1, \dots, P_s, G)$ .*

**Proof.** Using (3.14) and Lemma 10, we have

$$\dim_{\mathbb{F}_b}(\mathcal{N}_m) = ms - m \quad \text{and} \quad \dim_{\mathbb{F}_b}(\Xi_m) = m.$$

From (3.13), (4.68) and (4.69), we get that  $\mathcal{N}_m, \Xi_m \subset \mathbb{F}_b^{ms}$ .

By (2.17), in order to obtain the assertion of the lemma, it is sufficient to prove that  $A \cdot B = 0$  for all  $A \in \mathcal{N}_m$  and  $B \in \Xi_m$ .

According to (3.11), (3.13), (4.54) and (4.64) - (4.69), it is enough to verify that

$$A \cdot B = \sum_{i=1}^s \check{\theta}_i = 0 \quad \text{with} \quad \check{\theta}_i = \sum_{j=1}^m \theta_{i,j}(f) \theta_{i,j}^\perp((f^\perp, \varphi)) \quad \text{for all } f \in \mathcal{L}(G), \quad (4.76)$$

and  $(f^\perp, \varphi) \in (\mathcal{L}(G^\perp), \Phi)$ . From (4.54) and (4.62) - (4.64), we derive

$$\check{\theta}_i = \sum_{\check{j}_i=0}^{m_i-1} \varkappa_{i,\check{j}_i} \quad \text{with} \quad \varkappa_{i,\check{j}_i} = \sum_{\hat{j}_i=1}^{e_i} \theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}(f) \theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}^\perp((f^\perp, \varphi)). \quad (4.77)$$

Using (4.54) and (4.64)-(4.67), we have for  $\check{j}_i \in [0, m_i - 1]$ ,  $\hat{j}_i \in [1, e_i]$

$$\theta_{s,r_s+\check{j}_s e_s+\hat{j}_s}(f) = \vartheta_{s,\hat{j}_s}(f_{s,m_s-\check{j}_s-1}) \quad \text{and} \quad \theta_{s,r_s+\check{j}_s e_s+\hat{j}_s}^\perp((f^\perp, \varphi)) = \vartheta_{s,\hat{j}_s}^\perp(f_{s,-m_s+\check{j}_s}^\perp),$$

$$\theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}(f) = \vartheta_{i,\hat{j}_i}(f_{i,-\check{j}_i-1}) \quad \text{and} \quad \theta_{i,r_i+\check{j}_i e_i+\hat{j}_i}^\perp((f^\perp, \varphi)) = \vartheta_{i,\hat{j}_i}^\perp(f_{i,\check{j}_i}^\perp), \quad 1 \leq i \leq s-1.$$

By Lemma 8 and (4.77), we obtain

$$\varkappa_{s,\check{j}_s} = \sum_{\hat{j}_i=s}^{e_s} \vartheta_{s,\hat{j}_s}(f_{s,m_s-\check{j}_s-1}) \vartheta_{s,\hat{j}_s}^\perp(f_{s,-m_s+\check{j}_s}^\perp) = \text{Tr}_{F_{P_s}/\mathbb{F}_b}(f_{s,m_s-\check{j}_s-1} f_{s,-m_s+\check{j}_s}^\perp)$$

and

$$\varkappa_{i,\check{j}_i} = \sum_{\hat{j}_i=1}^{e_i} \vartheta_{i,\hat{j}_i}(f_{i,-\check{j}_i-1}) \vartheta_{i,\hat{j}_i}^\perp(f_{i,\check{j}_i}^\perp) = \text{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{i,-\check{j}_i-1} f_{i,\check{j}_i}^\perp) \quad \text{for } 1 \leq i \leq s-1.$$

From (4.74), (4.75) and (4.77), we get

$$\check{\mathfrak{d}}_i = \text{Res}_{P_i}(f f^\perp dt_s) \quad \text{for } 1 \leq i \leq s.$$

Applying Lemma 11, we get assertion (4.76). Hence Lemma 12 is proved.  $\square$

Let

$$G_i = \tilde{G} + q_i P_i - q_s P_s \text{ with } q_s = \left\lfloor \frac{g + \tilde{r}_0}{e_s} \right\rfloor + 1 \quad \text{and} \quad q_i = \left\lfloor \frac{g - \tilde{r}_0 + q_s e_s}{e_i} \right\rfloor + 1 \quad (4.78)$$

for  $i \in [1, s-1]$ . By (4.58), we have  $\deg(\tilde{G}) = g - 1 + \tilde{r}_0$  and  $\nu_{P_i}(\tilde{G}) = 0$ ,  $i \in [1, s]$ . It is easy to see that  $\deg(G_i) \geq 2g - 1$ ,  $i \in [1, s-1]$ . Let  $z_i = \dim(\mathcal{L}(G_i))$ , and let  $u_1^{(i)}, \dots, u_{z_i}^{(i)}$  be a basis of  $\mathcal{L}(G_i)$  over  $\mathbb{F}_b$ ,  $i \in [1, s-1]$ .

For each  $i \in [1, s-1]$ , we consider the chain

$$\mathcal{L}(G_i) \subset \mathcal{L}(G_i + P_i) \subset \mathcal{L}(G_i + 2P_i) \subset \dots$$

of vector spaces over  $\mathbb{F}_b$ . By starting from the basis  $u_1^{(i)}, \dots, u_{z_i}^{(i)}$  of  $\mathcal{L}(G_i)$  and successively adding basis vectors at each step of the chain, we obtain for each  $n \geq q_i$  a basis

$$\{u_1^{(i)}, \dots, u_{z_i}^{(i)}, k_{q_i,1}^{(i)}, \dots, k_{q_i,e_i}^{(i)}, \dots, k_{n,1}^{(i)}, \dots, k_{n,e_i}^{(i)}\} \quad (4.79)$$

of  $\mathcal{L}(G_i + (n - q_i + 1)P_i)$ . We note that we then have

$$k_{j_1,j_2}^{(i)} \in \mathcal{L}(G_i + (j_1 - q_i + 1)P_i) \text{ and } \nu_{P_i}(k_{j_1,j_2}^{(i)}) = -j_1 - 1, \quad \nu_{P_s}(k_{j_1,j_2}^{(i)}) \geq q_s \quad (4.80)$$

for  $j_1 \geq q_i$ ,  $1 \leq j_2 \leq e_i$ ,  $1 \leq i \leq s-1$ .

Let  $\check{G} = \tilde{G} + g P_s$ . We see that  $\deg(\check{G}) = g - 1 + \tilde{r}_0 + g e_s \geq 2g - 1$ . Let  $u_1^{(0)}, \dots, u_{z_0}^{(0)}$  be a basis of  $\mathcal{L}(\check{G})$  over  $\mathbb{F}_b$ . In a similar way, we construct a basis  $\{u_1^{(0)}, \dots, u_{z_0}^{(0)}, k_{0,1}^{(i)}, \dots, k_{0,e_i}^{(i)}, \dots, k_{(q_i-1),1}^{(i)}, \dots, k_{(q_i-1),e_i}^{(i)}\}$  of  $\mathcal{L}(\check{G} + q_i P_i)$  with

$$k_{j_1,j_2}^{(i)} \in \mathcal{L}(\check{G} + (j_1 + 1)P_i) \text{ and } \nu_{P_i}(k_{j_1,j_2}^{(i)}) = -j_1 - 1 \text{ for } j_1 \in [0, q_i), \quad (4.81)$$

$1 \leq j_2 \leq e_i$ ,  $1 \leq i \leq s-1$ .

Now, consider the chain

$$\mathcal{L}(q_s P_s - \tilde{G} + W) \subset \mathcal{L}((q_s + 1)P_s - \tilde{G} + W) \subset \dots \subset \mathcal{L}(G^\perp - P_s) \subset \mathcal{L}(G^\perp),$$

where  $G^\perp = m_s P_s - \tilde{G} + W$  and  $q_s = [(g + \tilde{r}_0)/e_s] + 1$ . By (4.57) and (4.58), we have  $\deg(\tilde{G}) = g - 1 + \tilde{r}_0$ ,  $\deg(W) = 2g - 2$  and  $\nu_{P_s}(\tilde{G}) = \nu_{P_s}(W) = 0$ . Hence  $\deg(q_s P_s - \tilde{G} + W) \geq 2g - 1$ . Let  $u_1^{(s)}, \dots, u_{z_s}^{(s)}$  be a basis of  $\mathcal{L}(q_s P_s - \tilde{G} + W)$  over  $\mathbb{F}_b$ . In a similar way, we construct a basis  $\{u_1^{(s)}, \dots, u_{z_s}^{(s)}, k_{q_s,1}^{(s)}, \dots, k_{q_s,e_s}^{(s)}, \dots, k_{n,1}^{(s)}, \dots, k_{n,e_s}^{(i)}\}$  of  $\mathcal{L}((n+1)P_s - \tilde{G} + W)$  with

$$k_{j_1,j_2}^{(s)} \in \mathcal{L}((j_1 + 1)P_s - \tilde{G} + W) \text{ and } \nu_{P_s}(k_{j_1,j_2}^{(s)}) = -j_1 - 1 \text{ for } j_1 \geq q_s \quad (4.82)$$

and  $j_2 \in [1, e_s]$ . By (4.79)-(4.81), we have the following local expansions

$$k_{j_1,j_2}^{(i)} := \sum_{r=-j_1}^{\infty} \chi_{j_1,r}^{(i,j_2)} t_i^{r-1} \quad \text{for } \chi_{j_1,r}^{(i,j_2)} \in F_{P_i}, \quad i \in [1, s]. \quad (4.83)$$

**Lemma 13.** *Let  $j_i \geq 0$  for  $i \in [1, s-1]$  and let  $j_s \geq q_s$ . Then  $\{\chi_{j_i,-j_i}^{(i,1)}, \dots, \chi_{j_i,-j_i}^{(i,e_i)}\}$  is a basis of  $F_{P_i}$  over  $\mathbb{F}_b$  for  $i \in [1, s]$ .*

**Proof.** Let  $i \in [1, s-1]$  and let  $j_i \geq q_i$ . Suppose that there exist  $a_1, \dots, a_{e_i} \in \mathbb{F}_b$ , such that  $\sum_{1 \leq j \leq e_i} a_j \chi_{j_i,-j_i}^{(i,j)} = 0$  and  $(a_1, \dots, a_{e_i}) \neq (\bar{0}, \dots, \bar{0})$ . By (4.83), we get  $\nu_{P_i}(\alpha) \geq -j_i$ , where  $\alpha := \sum_{1 \leq j \leq e_i} a_j k_{j_i,j_2}^{(i)}$ . Hence  $\alpha \in \mathcal{L}(G_i + (j_i - q_i)P_i)$ . We have a contradiction with the construction of the basis vectors (4.79).

Similarly, we can consider the cases  $i \in [1, s-1]$ ,  $j_i \in [0, q_i - 1]$  and  $i = s$ . Therefore Lemma 13 is proved.  $\square$

**Lemma 14.** *Let  $d_i \geq 1$  be an integer ( $i = 1, \dots, s-1$ ) and  $f^\perp \in G^\perp$ . Suppose that  $\text{Res}_{P_s, t_s}(f^\perp k_{j_1,j_2}^{(i)}) = 0$  for  $j_1 \in [0, d_i - 1]$ ,  $j_2 \in [1, e_i]$  and  $i \in [1, s-1]$ . Then*

$$\vartheta_{i,j_2}^\perp(f_{i,j_1}^\perp) = 0 \quad \text{for } j_1 \in [0, d_i - 1], j_2 \in [1, e_i] \text{ and } i \in [1, s-1]. \quad (4.84)$$

**Proof.** By (4.71), (4.72), (4.78), (4.80) and (4.81), we have  $\nu_P(\text{div}(f^\perp) + m_s P_s - \tilde{G} + W) \geq 0$ , for all  $P \in \mathbb{P}_F$  and  $k_{j_1,j_2}^{(i)} \in \mathcal{L}(\tilde{G} + a_{j_1} P_s + (j_1 + 1)P_i)$  with



some integer  $a_{j_1}$ .

From (2.4), (2.6) and (2.7), we derive

$$\nu_P(f^\perp k_{j_1, j_2}^{(i)} dt_s) \geq 0 \quad \text{and} \quad \text{Res}_P(f^\perp k_{j_1, j_2}^{(i)} dt_s) = 0 \quad \text{for all } P \in \mathbb{P}_F \setminus \{P_i, P_s\}.$$

Applying (4.60) and the Residue Theorem, we get

$$\text{Res}_{P_i, t_i}(f^\perp \tau_i k_{j_1, j_2}^{(i)}) = \text{Res}_{P_i}(f^\perp k_{j_1, j_2}^{(i)} dt_s) = -\text{Res}_{P_s}(f^\perp k_{j_1, j_2}^{(i)} dt_s) = -\text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)})$$

for all  $0 \leq j_1, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1$ .

By (4.61), (4.83) and the conditions of the lemma, we obtain

$$\begin{aligned} -\text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)}) &= \text{Res}_{P_i, t_i}(f^\perp \tau_i k_{j_1, j_2}^{(i)}) = \text{Res}_{P_i, t_i} \left( \sum_{j=0}^{\infty} f_{i,j}^\perp t_i^j \sum_{r=-j_1}^{\infty} \varkappa_{j_1, r}^{(i, j_2)} t_i^{r-1} \right) \\ &= \sum_{j=0}^{\infty} \sum_{r=-j_1}^{\infty} \text{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{i,j}^\perp \varkappa_{j_1, r}^{(i, j_2)}) \delta_{j, -r} = \sum_{j=0}^{j_1} \text{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{i,j}^\perp \varkappa_{j_1, -j}^{(i, j_2)}) = 0 \end{aligned} \quad (4.85)$$

for  $0 \leq j_1 \leq d_i - 1, 1 \leq j_2 \leq e_i$ , and  $1 \leq i \leq s-1$ .

Consider (4.85) for  $j_1 = 0$ . We have  $\text{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{i,0}^\perp \varkappa_{0,0}^{(i, j_2)}) = 0$  for all  $j_2 \in [1, e_i]$ . By Lemma 13, we obtain that  $f_{i,0}^\perp = 0$ . Suppose that  $f_{i,j}^\perp = 0$  for  $0 \leq j < j_0$ . Consider (4.85) for  $j_1 = j_0$ . We get  $\text{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{i,j_0}^\perp \varkappa_{j_0, -j_0}^{(i, j_2)}) = 0$  for all  $j_2 \in [1, e_i]$ . Applying Lemma 13, we have that  $f_{i,j_0}^\perp = 0$ . By induction, we obtain that  $f_{i,j}^\perp = 0$  for all  $j \in [0, d_i - 1]$  and  $i \in [1, s-1]$ . Now, using (4.62), we get that assertion (4.84) is true. Hence Lemma 14 is proved.  $\square$

**Lemma 15.** *Let  $s \geq 3$ ,  $\{\beta_{s,1}^\perp, \dots, \beta_{s,e_s}^\perp\}$  be a basis of  $F_{P_s}/\mathbb{F}_b$ ,*

$$\begin{aligned} \Lambda_1 &= \left\{ \left( \text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)}) \right)_{d_{i,1} \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1}, \right. \\ &\quad \left. \left( \text{Res}_{P_s, t_s}(\beta_{s,j_2}^\perp f^\perp t_s^{m_s - j_1 - 1}) \right)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid f^\perp \in \mathcal{L}(G^\perp) \right\} \end{aligned}$$

with  $d_{s,1} = m_s + 1 - [t/e_s] - (s-1)d_0\dot{m}e/e_s$ ,  $\dot{m} = [\tilde{m}\epsilon]$ ,  $\tilde{m} = m - r_0$ ,

$$d_{s,2} = m_s - 2 - [t/e_s] - (s-2)d_0\dot{m}e/e_s, \quad d_{i,1} = q_i, \quad d_{i,2} = d_0\dot{m}e/e_i - 1, \quad (4.86)$$

$i \in [1, s-1]$ ,  $d_0 = d+t$ ,  $e = e_1 e_2 \cdots e_s$ ,  $\epsilon = \eta(2(s-1)d_0 e)^{-1}$ ,  $\eta = (1 + \deg((t_s)_\infty))^{-1}$ .  
Then

$$\Lambda_1 = \mathbb{F}_b^\chi, \text{ with } \chi = \sum_{i=1}^s (d_{i,2} - d_{i,1} + 1) e_i \text{ for } m > 2(g-1+e_0)e_s + 2t(\eta^{-1}-1). \quad (4.87)$$

**Proof.** Suppose that (4.87) is not true. Then there exists  $b_{j_1, j_2}^{(i)} \in \mathbb{F}_b$  ( $i, j_1, j_2 \geq 1$ ) such that

$$\sum_{i=1}^s \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b_{j_1, j_2}^{(i)}| > 0 \quad (4.88)$$

and

$$\sum_{i=1}^{s-1} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b_{j_1, j_2}^{(i)} \text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)}) + \sum_{j_1=d_{s,1}}^{d_{s,2}} \sum_{j_2=1}^{e_s} b_{j_1, j_2}^{(s)} \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f^\perp t_s^{m_s - j_1 - 1}) = 0 \quad (4.89)$$

for all  $f^\perp \in \mathcal{L}(G^\perp)$ . Let  $\alpha = \alpha_1 + \alpha_2$  with

$$\alpha_1 = \sum_{i=1}^{s-1} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b_{j_1, j_2}^{(i)} k_{j_1, j_2}^{(i)} \quad \text{and} \quad \alpha_2 = \sum_{j_1=d_{s,1}}^{d_{s,2}} \sum_{j_2=1}^{e_s} b_{j_1, j_2}^{(s)} \beta_{s, j_2}^\perp t_s^{m_s - j_1 - 1}. \quad (4.90)$$

By (4.89), we have

$$\text{Res}_{P_s, t_s}(f^\perp \alpha) = 0 \quad \text{for all } f^\perp \in \mathcal{L}(G^\perp). \quad (4.91)$$

From (4.80), we get  $\nu_{P_s}(\alpha) \geq q_s$ . Consider the local expansion

$$\alpha = \sum_{r=q_s}^{\infty} \varphi_r t_s^r \quad \text{with } \varphi_r \in F_{P_s} \quad \text{for } r \geq q_s.$$

Suppose that  $m_s > j_0 := \nu_{P_s}(\alpha)$ . Therefore  $\varphi_{j_0} \neq 0$ . From (4.82), we obtain that  $k_{j_0, j_2}^{(s)} \in \mathcal{L}(G^\perp)$  for all  $j_2 \in [1, e_s]$ . Applying (4.83) and (4.91), we derive

$$\text{Res}_{P_s, t_s}(k_{j_0, j_2}^{(s)} \alpha) = \text{Res}_{P_s, t_s} \left( \sum_{j=-j_0}^{\infty} \varkappa_{j_0, j}^{(s, j_2)} t_s^{j-1} \sum_{r=j_0}^{\infty} \varphi_r t_s^r \right) = \text{Tr}_{F_{P_s}/\mathbb{F}_b}(\varkappa_{j_0, -j_0}^{(s, j_2)} \varphi_{j_0}) = 0$$

for all  $j_2 \in [1, e_s]$ . By Lemma 13,  $\{\mathfrak{x}_{j_0, -j_0}^{(s,1)}, \dots, \mathfrak{x}_{j_0, -j_0}^{(s, e_s)}\}$  is a basis of  $F_{P_s}$ . Hence  $\varphi_{j_0} = 0$ . We have a contradiction. Thus  $\nu_{P_s}(\alpha) \geq m_s$ .

We consider the compositum field  $F' = FF_{P_s}$ . Let  $\mathfrak{B}_1, \dots, \mathfrak{B}_\mu$  be all the places of  $F'/F_{P_s}$  lying over  $P_s$ . From (2.11), we get

$$\nu_{\mathfrak{B}_i}(\alpha) \geq m_s \quad \text{for } i = 1, \dots, \mu. \quad (4.92)$$

According to (4.78) and (4.80), we obtain

$$\alpha_1 \in \mathcal{L}_F(A_1) = \mathcal{L}(A_1), \quad \text{with } A_1 := \tilde{G} - q_s P_s + \sum_{i=1}^{s-1} (d_{i,2} + 1) P_i.$$

Applying Theorem D(d), we have

$$\alpha_1 \in \mathcal{L}_{F'}(\text{Con}_{F'/F}(A_1)).$$

By (4.90), we derive

$$\alpha_2 \in \mathcal{L}_{F'}(A_2), \quad \text{with } A_2 = ((t_s)_\infty^{F'})^{m_s - d_{s,1} - 1}.$$

Using (4.92), we get

$$\alpha \in \mathcal{L}_{F'}(A_1 + A_2 - m_s \sum_{i=1}^{\mu} \mathfrak{B}_i).$$

From (2.9), Theorem D(a) and Theorem E, we derive  $\text{Con}_{F'/F}(P_s) = \sum_{i=1}^{\mu} \mathfrak{B}_i$ ,  $\text{Con}_{F'/F}((t_s)_\infty^F) = (t_s)_\infty^{F'}$  and

$$\alpha \in \mathcal{L}_{F'}(A_3), \quad \text{with } A_3 = \text{Con}_{F'/F}(A_1 + (m_s - d_{s,1} - 1)(t_s)_\infty^F - m_s P_s).$$

Applying Theorem D(c) and (4.78), we have

$$\begin{aligned} \deg(A_3) &= \deg\left(\tilde{G} + \sum_{i=1}^{s-1} (d_{i,2} + 1) P_i + (m_s - d_{s,1} - 1)(t_s)_\infty^F - m_s P_s\right) \\ &\leq g - 1 + \tilde{r}_0 + (s - 1)d_0 e \dot{m} + (m_s - d_{s,1} - 1)\deg((t_s)_\infty) - m_s e_s \\ &\leq g - 1 + e_0 - e_s + (s - 1)d_0 e \dot{m} + ([t/e_s] + (s - 1)d_0 \dot{m} e / e_s - 2)(\eta^{-1} - 1) \\ &\quad - m_s e_s \leq g - 1 + e_0 + (t/e_s - 2)(\eta^{-1} - 1) + (s - 1)d_0 e \dot{m}(1 + (\eta^{-1} - 1)/e_s) - m \\ &\leq g - 1 + e_0 + t(\eta^{-1} - 1)/e_s - m((2e_s)^{-1} + (1 - \eta/2)(1 - 1/e_s)) \leq \beta - m/(2e_s) < 0 \end{aligned}$$

for  $m > 2e_s\beta$ , with  $\beta = g - 1 + e_0 + t(\eta^{-1} - 1)/e_s$  and  $\epsilon = \eta(2(s - 1)d_0 e)^{-1}$ .

Hence  $\alpha = 0$ .

Suppose that  $\sum_{i=1}^{s-1} \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b_{j_1,j_2}^{(i)}| = 0$ . Then  $\alpha_2 = 0$  and  $\sum_{j_2=1}^{e_s} b_{j_1,j_2}^{(s)} \beta_{s,j_2}^\perp = 0$  for all  $j_1 \in [d_{s,1}, d_{s,2}]$ . Bearing in mind that  $(\beta_{s,j_2}^\perp)_{1 \leq j_2 \leq e_s}$  is a basis of  $F_{P_s}/\mathbb{F}_b$ , we get  $\sum_{j_1=d_{s,1}}^{d_{s,2}} \sum_{j_2=1}^{e_s} |b_{j_1,j_2}^{(s)}| = 0$ . By (4.88), we have a contradiction.

Therefore there exists  $h \in [1, s-1]$  with

$$\sum_{j_1=d_{h,1}}^{d_{h,2}} \sum_{j_2=1}^{e_h} |b_{j_1,j_2}^{(h)}| > 0. \quad (4.93)$$

Let  $\mathfrak{B}_{h,1}, \dots, \mathfrak{B}_{h,\mu_h}$  be all the places of  $F'/F_{P_s}$  lying over  $P_h$ . Let

$$\alpha_{1,i} = \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b_{j_1,j_2}^{(i)} k_{j_1,j_2}^{(i)}, \quad i = 1, \dots, s-1.$$

Let  $\nu_{P_h}(t_s) \geq 0$  or  $\alpha_2 = 0$ . Therefore  $\nu_{\mathfrak{B}_{h,j}}(\alpha_2) \geq 0$  for  $1 \leq j \leq \mu_h$ . Taking into account that  $\alpha_1 = -\alpha_2$ , we get  $\nu_{\mathfrak{B}_{h,j}}(\alpha_1) \geq 0$  for  $1 \leq j \leq \mu_h$ , and  $\nu_{P_h}(\alpha_1) \geq 0$ . Using (4.58), (4.78), (4.80) and (4.86), we obtain  $\nu_{P_h}(\alpha_{1,i}) \geq 0$  for  $1 \leq i \leq s-1, i \neq h$ . Bearing in mind (4.93) and that  $\{u_1^{(h)}, \dots, u_{z_h}^{(h)}, k_{q_h,1}^{(h)}, \dots, k_{q_h,e_h}^{(h)}, \dots, k_{n,1}^{(h)}, \dots, k_{n,e_h}^{(h)}\}$  is a basis of  $\mathcal{L}(G_h + (n - q_h + 1)P_h)$ , we get

$$\alpha_{1,h} \in \mathcal{L}(G_h + (j - q_h + 1)P_h) \setminus \mathcal{L}(G_h + (j - q_h)P_h) \text{ with some } j \geq q_h.$$

By (4.78) and (4.80), we get  $\nu_{P_h}(\alpha_{1,h}) \leq -1$ . We have a contradiction.

Now let  $\nu_{P_h}(t_s) \leq -1$  and  $\alpha_2 \neq 0$ . We have  $\nu_{P_h}(\alpha_{1,h}) \geq -d_{h,2} - 1$ ,  $\nu_{P_h}(\alpha_1) \geq -d_{h,2} - 1$  and  $\nu_{\mathfrak{B}_{h,j}}(\alpha_1) \geq -d_{h,2} - 1$ ,  $j = 1, \dots, \mu_h$ . On the other hand, using (4.90) and (2.11), we have  $\nu_{\mathfrak{B}_{h,j}}(\alpha_2) \leq -(m_s - d_{s,2} - 1)$ ,  $j = 1, \dots, \mu_h$ . According to (3.17) and (4.86), we obtain  $s \geq 3$ ,  $e_h \geq e_s$  and

$$m_s - d_{s,2} - 1 - d_{h,2} - 1 = [t/e_s] + 1 + (s-2)d_0 e \dot{m} e / e_s - d_0 \dot{m} e / e_h \geq 1.$$

We have a contradiction. Thus assertion (4.89) is not true. Hence (4.87) is true and Lemma 15 follows.  $\square$

### End of the proof of Theorem 3.

Using (2.15), (3.15), (4.67)-(4.69) and Lemma 12, we have

$$\mathcal{P}_1 = \{\tilde{x}(f^\perp, \varphi) = (\tilde{x}_1(f^\perp, \varphi), \dots, \tilde{x}_s(f^\perp, \varphi)) \mid f^\perp \in \mathcal{L}(G^\perp), \varphi \in \Phi\} \quad (4.94)$$

with

$$\tilde{x}_i(f^\perp, \varphi) = \sum_{j=1}^m \phi^{-1}(\theta_{i,j}^\perp(f^\perp, \varphi))b^{-j} = \sum_{j=1}^{r_i} \phi^{-1}(\varphi_{i,j})b^{-j} + b^{-r_i} \sum_{j=1}^{m-r_i} \phi^{-1}(\dot{\theta}_{i,j}^\perp(f^\perp))b^{-j}.$$

By (3.16), we have

$$\mathcal{P}_2 = \{\dot{\mathbf{x}}(f^\perp) = (\dot{x}_1(f^\perp), \dots, \dot{x}_s(f^\perp)) \mid f^\perp \in \mathcal{L}(G^\perp)\} \quad (4.95)$$

with

$$\dot{x}_i(f^\perp) = \sum_{j=1}^{m-r_i} \phi^{-1}(\dot{\theta}_{i,j}^\perp(f^\perp))b^{-j}, \quad 1 \leq i \leq s. \quad (4.96)$$

**Lemma 16.** *With notation as above,  $\mathcal{P}_2$  is a  $d$ -admissible  $(t, m - r_0, s)$  net in base  $b$  with  $d = g + e_0$ , and  $t = g + e_0 - s$ .*

**Proof.** Let  $J = \prod_{i=1}^s [A_i/b^{d_i}, (A_i + 1)/b^{d_i}]$  with  $d_i \geq 0$ , and  $0 \leq A_i < b^{d_i}$ ,  $1 \leq i \leq s$ , and let  $J_\psi = \prod_{i=1}^s [\psi_i/b^{r_i} + A_i/b^{r_i+d_i}, \psi_i/b^{r_i} + (A_i + 1)/b^{r_i+d_i}]$  with  $\psi_i/b^{r_i} = \psi_{i,1}/b + \dots + \psi_{i,r_i}/b^{r_i}$ ,  $\psi_{i,j} \in Z_b$ ,  $1 \leq i \leq s$ ,  $d_1 + \dots + d_s = m - r_0 - t$ . It is easy to see, that

$$\dot{\mathbf{x}}(f^\perp) \in J \iff \tilde{\mathbf{x}}(f^\perp, \varphi) \in J_\psi \quad \text{with} \quad \psi_{i,j} = \phi^{-1}(\varphi_{i,j}), \quad 1 \leq j \leq r_i, \quad 1 \leq i \leq s.$$

Bearing in mind that  $\mathcal{P}_1$  is a  $(t, m, s)$  net with  $t = g + e_0 - s$ , we have

$$\sum_{f^\perp \in \mathcal{L}(G^\perp)} \mathbf{1}(J, \dot{\mathbf{x}}(f^\perp)) = \sum_{f^\perp \in \mathcal{L}(G^\perp), \varphi \in \Phi} \mathbf{1}(J_\psi, \mathbf{x}(f^\perp, \varphi)) = b^t.$$

Therefore  $\mathcal{P}_2$  is a  $(t, m - r_0, s)$  net in base  $b$  with  $t = g + e_0 - s$ .

Using (4.69), Definition 5 and Definition 9, we can get  $d$  from the following equation  $-\delta_m^\perp(\dot{\Xi}_m) = -(m - r_0) - d + 1$ . Applying Lemma 9, we obtain  $-(m + g - 1 + e_0 - r_0) \leq -(m - r_0) - d + 1$ . Hence  $d \leq g + e_0$ . Thus Lemma 16 is proved.  $\square$

Let  $V_i \subseteq \mathbb{F}_b^{\mu_i}$  be a vector space over  $\mathbb{F}_b$ ,  $\mu_i \geq 1$ ,  $i = 1, 2$ . Consider a linear map  $h : V_1 \rightarrow V_2$ . By the first isomorphism theorem, we have

$$\dim_{\mathbb{F}_b}(V_1) = \dim_{\mathbb{F}_b}(\ker(h)) + \dim_{\mathbb{F}_b}(\text{im}(h)). \quad (4.97)$$

Let

$$\Lambda_1' = \left\{ \left( \text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)}) \right)_{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1}, \right. \\ \left. \left( \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f^\perp t_s^{m_s - j_1 - 1}) \right)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid f^\perp \in \mathcal{L}(G^\perp) \right\}$$

and

$$\Lambda_2 = \left\{ \left( \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f^\perp t_s^{m_s - j_1 - 1}) \right)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid \text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)}) = 0 \right. \\ \left. \text{for } 0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1, f^\perp \in \mathcal{L}(G^\perp) \right\}$$

with  $d_{s,1} = m_s + 1 - [t/e_s] - (s-1)d_0\dot{m}e/e_s$ ,

$$d_{s,2} = m_s - 2 - [t/e_s] - (s-2)d_0\dot{m}e/e_s, \quad d_{i,1} = q_i, \quad d_{i,2} = d_0\dot{m}e/e_i - 1, \quad (4.98)$$

$i \in [1, s-1]$ ,  $d_0 = d+t$ ,  $e = e_1 e_2 \cdots e_s$ ,  $\epsilon = \eta(2(s-1)d_0e)^{-1}$ ,  $\eta = (1 + \deg((t_s)_\infty))^{-1}$ ,  $\dot{m} = [\tilde{m}\epsilon]$ ,  $\tilde{m} = m - r_0$ ,  $m > 2(g-1+e_0)e_s + 2t(\eta^{-1}-1)$ ,  $d = g+e_0$  and  $t = g+e_0-s$ .

By (4.97), (4.98) and Lemma 15, we have  $\dim_{\mathbb{F}_b}(\Lambda_1') \geq \dim_{\mathbb{F}_b}(\Lambda_1)$  and

$$\dim_{\mathbb{F}_b}(\Lambda_2) = \dim_{\mathbb{F}_b}(\Lambda_1') - \dim_{\mathbb{F}_b} \left( \left\{ \left( \text{Res}_{P_s, t_s}(f^\perp k_{j_1, j_2}^{(i)}) \right)_{\substack{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i \\ 1 \leq i \leq s-1}} \mid f^\perp \in \mathcal{L}(G^\perp) \right\} \right) \\ \geq \dim_{\mathbb{F}_b}(\Lambda_1) - \sum_{i=1}^{s-1} (d_{i,2} + 1)e_i \geq (d_{s,2} - d_{s,1} + 1)e_s - \sum_{i=1}^{s-1} q_i e_i = d_0\dot{m}e - 2e_s - \sum_{i=1}^{s-1} q_i e_i.$$

Let

$$\Lambda_3 = \left\{ \left( \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f^\perp t_s^{m_s - j_1 - 1}) \right)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s} \mid \vartheta_{i, j_2}^\perp(f_{i, j_1}^\perp) = 0 \right. \\ \left. \text{for } 0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1 \mid f^\perp \in \mathcal{L}(G^\perp) \right\}.$$

Using Lemma 14, we get  $\Lambda_3 \supseteq \Lambda_2$  and  $\dim_{\mathbb{F}_b}(\Lambda_3) \geq \dim_{\mathbb{F}_b}(\Lambda_2)$ . Let

$$\Lambda_4 = \left\{ \left( \vartheta_{i, j_2}^\perp(f_{i, j_1}^\perp) \right)_{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1} \mid f^\perp \in \mathcal{L}(G^\perp) \right\}.$$

Taking into account that  $\mathcal{P}_2$  is a  $(t, m - r_0, s)$  net in base  $b$ , we get from (4.95) that  $\dim_{\mathbb{F}_b}(\Lambda_4) = (s-1)d_0\dot{m}e$ . Let

$$\Lambda_5 = \left\{ \left( \vartheta_{i, j_2}^\perp(f_{i, j_1}^\perp) \right)_{\substack{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i \\ 1 \leq i \leq s-1}}, \left( \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f^\perp t_s^{m_s - j_1 - 1}) \right)_{\substack{d_{s,1} \leq j_1 \leq d_{s,2} \\ 1 \leq j_2 \leq e_s}} \mid f^\perp \in \mathcal{L}(G^\perp) \right\}.$$

By (4.78) and (4.97), we have

$$\dim_{\mathbb{F}_b}(\Lambda_5) = \dim_{\mathbb{F}_b}(\Lambda_3) + \dim_{\mathbb{F}_b}(\Lambda_4) \geq sd_0em - 2e_s - 2(s-1)(g+e_0).$$

Let  $\dot{m}_1 = d_0em$ ,  $\dot{m} = [\tilde{m}\epsilon]$ ,  $\ddot{m}_i = 0$ ,  $i \in [1, s-1]$  and  $\ddot{m}_s = m - t - (s-1)\dot{m}_1$ . Bearing in mind that  $\theta_{i, \hat{j}_i e_i + \hat{j}_i}^\perp(f^\perp) = \vartheta_{i, \hat{j}_i}^\perp(f_{i, \hat{j}_i}^\perp)$  for  $1 \leq \hat{j}_i \leq e_i$ ,  $0 \leq \check{j}_i \leq m_i - 1$ ,  $i \in [1, s-1]$  (see (4.63)), we obtain

$$\left( \dot{\theta}_{i, \ddot{m}_i + j}^\perp(f^\perp) \right)_{1 \leq j \leq \dot{m}_1, 1 \leq i \leq s-1} \supseteq \left( \vartheta_{i, j_2}^\perp(f_{i, j_1}^\perp) \right)_{0 \leq j_1 \leq d_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s-1}. \quad (4.99)$$

From (4.98), we have  $\ddot{m}_s < d_{s,1}e_s$  and  $(d_{s,2}+1)e_s < \ddot{m}_s + \dot{m}_1$ . Taking into account that

$$\dot{\theta}_{s, j_1 e_s + j_2}^\perp(f_{s, -m_s + j_1}^\perp) = \vartheta_{s, j_2}^\perp(f^\perp) = \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f_s^\perp t_s^{m_s - j_1 - 1})$$

(see (4.62) and (4.64)), we get

$$\left( \dot{\theta}_{s, \ddot{m}_s + j}^\perp(f^\perp) \right)_{1 \leq j \leq \dot{m}_1} \supseteq \left( \text{Res}_{P_s, t_s}(\beta_{s, j_2}^\perp f_s^\perp t_s^{m_s - j_1 - 1}) \right)_{d_{s,1} \leq j_1 \leq d_{s,2}, 1 \leq j_2 \leq e_s}. \quad (4.100)$$

Let

$$\Lambda_6 = \left\{ \left( \left( \dot{\theta}_{i, \ddot{m}_i + j}^\perp(f^\perp) \right)_{1 \leq j \leq \dot{m}_1, 1 \leq i \leq s} \right) \mid f^\perp \in \mathcal{L}(G^\perp) \right\}.$$

By (4.99) and (4.100), we derive

$$\dim_{\mathbb{F}_b}(\Lambda_6) \geq \dim_{\mathbb{F}_b}(\Lambda_5) \geq sd_0em - 2e_s - 2(s-1)(g+e_0).$$

Applying (2.15), (3.16), (4.95) and Lemma 2, we get that there exists  $B_i \in \{0, \dots, \dot{m} - 1\}$ ,  $1 \leq i \leq s$  such that

$$\Lambda_7 = \mathbb{F}_b^{sd_0em - d_0eB} \quad \text{for } \dot{m} \geq 1, \quad (4.101)$$

where  $B = \#B_1 + \dots + \#B_s \leq 4(s-1)(g+e_0)$  and

$$\Lambda_7 = \left\{ \left( \dot{\theta}_{i, \ddot{m}_i + j_i d_0 e + \check{j}_i}^\perp(f^\perp) \mid \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, d_0 e], i \in [1, s] \right) \mid f^\perp \in \mathcal{L}(G^\perp) \right\}$$

with  $\bar{B}_i = \{0, \dots, \dot{m} - 1\} \setminus B_i$ .

From (4.96), we have

$$\left\{ \left( \dot{x}_{i, \ddot{m}_i + j_i d_0 e + \check{j}_i}(f^\perp) \mid \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, d_0 e], i \in [1, s] \right) \mid f^\perp \in \mathcal{L}(G^\perp) \right\} = Z_b^{sd_0em - d_0eB}.$$

We apply Corollary 2 with  $\dot{s} = s$ ,  $\tilde{r} = r_0$ ,  $\tilde{m} = m - r_0$ ,  $\epsilon = \eta(2(s-1)d_0e)^{-1}$  and  $\hat{e} = e = e_1e_2 \cdots e_s$ .

Let  $\dot{\gamma}(f^\perp, \dot{\mathbf{w}}) = \dot{\gamma} = (\dot{\gamma}^{(1)}, \dots, \dot{\gamma}^{(s)})$  with  $\dot{\gamma}^{(i)} := [(\dot{\mathbf{x}}(f^\perp) \oplus \dot{\mathbf{w}})^{(i)}]_{\tilde{m}_i}$ ,  $i \in [1, s]$ . Using (4.96) and (4.101), we get that there exists  $f^\perp \in G^\perp$  such that  $\dot{\gamma}(f^\perp, \dot{\mathbf{w}})$  satisfy (2.25). Bearing in mind Lemma 16, we get from Corollary 2 that

$$\left| \Delta((\dot{\mathbf{x}}(f^\perp) \oplus \dot{\mathbf{w}})_{f^\perp \in G^\perp}, J_{\dot{\gamma}}) \right| \geq 2^{-2} b^{-d} K_{d,t,s}^{-s+1} \eta^{s-1} m^{s-1}. \quad (4.102)$$

for  $m \geq 2^{2s+3} b^{d+t+s} (d+t)^s (s-1)^{2s-1} (g+e_0) e \eta^{-s+1}$ .

Taking into account (1.2), and that  $\dot{\mathbf{w}} \in E_{m-r_0}^s$  is arbitrary, we get the second assertion in Theorem 3.

Consider the first assertion in Theorem 3.

Let  $\tilde{\gamma} = (\tilde{\gamma}^{(1)}, \dots, \tilde{\gamma}^{(s)})$  with  $\tilde{\gamma}^{(i)} = b^{-r_i} \dot{\gamma}^{(i)}$ ,  $i \in [1, s]$ , and let  $\tilde{\mathbf{w}} = (\tilde{w}^{(1)}, \dots, \tilde{w}^{(s)}) \in E_m^s$  with  $\tilde{w}_{j+r_i}^{(i)} = \dot{w}_j^{(i)}$  for  $j \in [1, m-r_0]$ ,  $i \in [1, s]$ . By (4.94) and (4.95), we have

$$\tilde{x}_i(f^\perp, \varphi) \oplus \tilde{w}^{(i)} \in [0, \tilde{\gamma}_i] \iff \dot{x}_i(f^\perp) \oplus \dot{w}^{(i)} \in [0, \dot{\gamma}_i] \text{ and } \phi^{-1}(\varphi_{i,j}) \oplus \tilde{w}_{i,j} = 0$$

for  $j \in [1, r_i]$ ,  $i \in [1, s]$ . Hence

$$\sum_{\varphi \in \Phi} (\mathbb{1}([0, \tilde{\gamma}], \tilde{\mathbf{x}}(f^\perp, \varphi) \oplus \tilde{\mathbf{w}}) - \tilde{\gamma}_0) = \mathbb{1}([0, \dot{\gamma}], \dot{\mathbf{x}}(f^\perp) \oplus \dot{\mathbf{w}}) - \dot{\gamma}_0,$$

where  $[0, \dot{\gamma}] = \prod_{i=1}^s [0, \dot{\gamma}^{(i)}]$ ,  $[0, \tilde{\gamma}] = \prod_{i=1}^s [0, \tilde{\gamma}^{(i)}]$ ,  $\tilde{\gamma}_0 = \tilde{\gamma}^{(1)} \dots \tilde{\gamma}^{(s)}$  and  $\dot{\gamma}_0 = \dot{\gamma}^{(1)} \dots \dot{\gamma}^{(s)}$ . Therefore

$$\sum_{f^\perp \in \mathcal{L}(G^\perp), \varphi \in \Phi} (\mathbb{1}([0, \tilde{\gamma}], \tilde{\mathbf{x}}(f^\perp, \varphi) \oplus \tilde{\mathbf{w}}) - \tilde{\gamma}_0) = \sum_{f^\perp \in \mathcal{L}(G^\perp)} (\mathbb{1}([0, \dot{\gamma}], \dot{\mathbf{x}}(f^\perp) \oplus \dot{\mathbf{w}}) - \dot{\gamma}_0).$$

Using (1.1), (1.2) and (4.102), we get the first assertion in Theorem 3.

Thus Theorem 3 is proved.  $\square$

#### 4.4 Halton-type sequences. Proof of Theorem 4.

Using (3.24) and (3.25), we define the sequence  $(\mathbf{x}_{n,j}^{(i)})_{j \geq 1}$  by

$$\sum_{j_2=1}^{e_i} x_{n,j_1 e_i + j_2}^{(i)} b^{-j_2 + e_i} := \sigma_{P_i}(f_{n,j_1}^{(i)}), \quad x_n^{(i)} := \sum_{j=0}^{\infty} \frac{x_{n,j}^{(i)}}{b^j} = \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{e_i} \frac{x_{n,j_1 e_i + j_2}^{(i)}}{b^{j_1 e_i + j_2}}, \quad (4.103)$$



$1 \leq i \leq s$ , with  $(x_n^{(1)}, \dots, x_n^{(s)}) = \mathbf{x}_n = \xi(f_n)$ , and  $n = 0, 1, \dots$ .

**Lemma 17.**  $(\mathbf{x}_n)_{n \geq 0}$  is  $d$ -admissible with  $d = g + e_0$ , where  $e_0 = e_1 + \dots + e_s$ .

**Proof.** Suppose that the assertion of the lemma is not true. By (1.4), there exists  $\dot{n} > \dot{k}$  such that  $\left\| \dot{n} \ominus \dot{k} \right\|_b \left\| \mathbf{x}_{\dot{n}} \ominus \mathbf{x}_{\dot{k}} \right\|_b < b^{-d}$ .

Let  $d_i + 1 = \dot{d}_i e_i + \ddot{d}_i$  with  $1 \leq \ddot{d}_i \leq e_i$ ,  $1 \leq i \leq s$ ,  $n = \dot{n} \ominus \dot{k}$ ,  $\|n\|_b = b^{m-1}$  and let  $\left\| \mathbf{x}_{\dot{n}}^{(i)} \ominus \mathbf{x}_{\dot{k}}^{(i)} \right\|_b = b^{-d_i-1}$ ,  $1 \leq i \leq s$ . Hence  $m - 1 - \sum_{i=1}^s (d_i + 1) \leq -d - 1$ , and

$$m + g - 1 - \sum_{i=1}^s \dot{d}_i e_i \leq m + g - 1 - \sum_{i=1}^s (d_i + 1) + e_0 \leq -d - 1 + g + e_0 < 0. \quad (4.104)$$

We have

$$a_{m-1}(n) \neq 0, \quad a_r(n) = 0, \quad \text{for } r \geq m, \quad x_{\dot{n}, d_i+1}^{(i)} \neq x_{\dot{k}, d_i+1}^{(i)}, \quad x_{\dot{n}, r}^{(i)} = x_{\dot{k}, r}^{(i)} \quad (4.105)$$

for  $r \leq d_i$ ,  $1 \leq i \leq s$ . From (4.103), we get

$$f_{\dot{n}, j_1}^{(i)} = f_{\dot{k}, j_1}^{(i)} \quad \text{and} \quad f_{\dot{n}, j_1}^{(i)} = 0 \quad \text{for} \quad 0 \leq j_1 < \dot{d}_i, \quad 1 \leq i \leq s.$$

Suppose that  $f_{\dot{n}, d_i}^{(i)} = 0$ , then  $f_{\dot{n}, d_i}^{(i)} = f_{\dot{k}, d_i}^{(i)}$  and  $x_{\dot{n}, j}^{(i)} = x_{\dot{k}, j}^{(i)}$  for  $1 \leq j \leq (\dot{d}_i + 1)e_i$ .

Taking into account that  $d_i + 1 \leq (\dot{d}_i + 1)e_i$ , we have a contradiction. Therefore  $f_{\dot{n}, d_i}^{(i)} \neq 0$ , for all  $1 \leq i \leq s$ . Applying (3.23), we derive  $\nu_{P_i}(f_n) = \dot{d}_i$ ,  $1 \leq i \leq s$ .

Using (3.18)-(3.20) and (4.105), we obtain  $f_n \in \mathcal{L}((m+g-1)P_{s+1} - \sum_{i=1}^s \dot{d}_i P_i) \setminus \{0\}$ . By (4.104), we get

$$\deg((m+g-1)P_{s+1} - \sum_{i=1}^s \dot{d}_i P_i) = m + g - 1 - \sum_{i=1}^s \dot{d}_i e_i < 0.$$

Hence  $f_n = 0$ . We have a contradiction. Thus Lemma 17 is proved.  $\square$

Consider the  $H$ -differential  $dt_{s+1}$ . By Proposition A, we have that there exists  $\tau_i$  with  $dt_{s+1} = \tau_i dt_i$ ,  $1 \leq i \leq s$ . Let  $W = \text{div}(dt_{s+1})$ , and let

$$G_i = W + q_i P_i - g P_{s+1}, \quad \text{with} \quad q_i = [(g+1)/e_i + 1], \quad 1 \leq i \leq s. \quad (4.106)$$

It is easy to see that  $\deg(G_i) \geq 2g - 2 + g + 1 - g = 2g - 1$ ,  $1 \leq i \leq s$ . Let  $z_i = \dim(\mathcal{L}(G_i))$ , and let  $u_1^{(i)}, \dots, u_{z_i}^{(i)}$  be a basis of  $\mathcal{L}(G_i)$  over  $\mathbb{F}_b$ ,  $1 \leq i \leq s$ . For each  $1 \leq i \leq s - 1$ , we consider the chain

$$\mathcal{L}(G_i) \subset \mathcal{L}(G_i + P_i) \subset \mathcal{L}(G_i + 2P_i) \subset \dots$$

of vector spaces over  $\mathbb{F}_b$ . By starting from the basis  $u_1^{(i)}, \dots, u_{z_i}^{(i)}$  of  $\mathcal{L}(G_i)$  and successively adding basis vectors at each step of the chain, we obtain for each  $n \geq q_i$  a basis

$$\{u_1^{(i)}, \dots, u_{z_i}^{(i)}, k_{q_i,1}^{(i)}, \dots, k_{q_i,e_i}^{(i)}, \dots, k_{n,1}^{(i)}, \dots, k_{n,e_i}^{(i)}\}$$

of  $\mathcal{L}(G_i + (n - q_i + 1)P_i)$ . We note that we then have

$$k_{j_1,j_2}^{(i)} \in \mathcal{L}(G_i + (j_1 - q_i + 1)P_i) \setminus \mathcal{L}(G_i + (j_1 - q_i)P_i) \quad (4.107)$$

for  $q_i \leq j_1$ ,  $1 \leq j_2 \leq e_i$  and  $1 \leq i \leq s$ . Hence

$$\operatorname{div}(k_{j_1,j_2}^{(i)}) + W - gP_{s+1} + (j_1 + 1)P_i \geq 0 \quad \text{and} \quad \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) + \nu_{P_{s+1}}(W) \geq g.$$

From (2.4) and (2.6), we obtain

$$\nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) = \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)} dt_{s+1}) = \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) + \nu_{P_{s+1}}(W).$$

Therefore

$$\nu_{P_{s+1}}(W) = 0 \quad \text{and} \quad \nu_{P_{s+1}}(k_{j_1,j_2}^{(i)}) \geq g. \quad (4.108)$$

Now, let  $\check{G}_i = W + (e_i + 1)P_{s+1} - P_i$ . We see that  $\deg(\check{G}_i) = 2g - 1$ . Let  $\check{u}_1^{(i)}, \dots, \check{u}_{\check{z}_i}^{(i)}$  be a basis of  $\mathcal{L}(\check{G}_i)$  over  $\mathbb{F}_b$ . In a similar way, we construct a basis  $\{\check{u}_1^{(i)}, \dots, \check{u}_{\check{z}_i}^{(i)}, k_{0,1}^{(i)}, \dots, k_{0,e_i}^{(i)}, \dots, k_{q_i-1,1}^{(i)}, \dots, k_{q_i-1,e_i}^{(i)}\}$  of  $\mathcal{L}(\check{G} + q_i P_i)$  with

$$k_{j_1,j_2}^{(i)} \in \mathcal{L}(\check{G} + (j_1 + 1)P_i) \setminus \mathcal{L}(\check{G} + j_1 P_i) \text{ for } j_1 \in [0, q_i], j_2 \in [1, e_i], i \in [1, s]. \quad (4.109)$$

**Lemma 18.** *Let  $\{\beta_1^{(i)}, \dots, \beta_{e_i}^{(i)}\}$  be a basis of  $F_{P_i}/\mathbb{F}_b$ ,  $s \geq 2$ ,  $d_i \geq 1$  be integer ( $i = 1, \dots, s$ ) and  $n \in [0, b^m]$ . Suppose that  $\operatorname{Res}_{P_{s+1}, t_{s+1}}(f_n k_{j_1,j_2}^{(i)}) = 0$  for  $j_1 \in [0, d_i - 1]$ ,  $j_2 \in [1, e_i]$  and  $i \in [1, s]$ . Then*

$$\operatorname{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) = 0 \quad \text{for } j_1 \in [0, d_i - 1], j_2 \in [1, e_i] \text{ and } i \in [1, s].$$

**Proof.** Using (4.107) and (4.109), we get

$$\nu_{P_i}(k_{j_1, j_2}^{(i)}) = -j_1 - 1 - \nu_{P_i}(W) \quad \text{for } j_1 \geq 0, j_2 \in [1, e_i] \text{ and } i \in [1, s].$$

From (2.4) and (2.6), we obtain

$$\nu_{P_i}(\tau_i) = \nu_{P_i}(\tau_i dt_i) = \nu_{P_i}(dt_{s+1}) = \nu_{P_i}(\operatorname{div}(dt_{s+1})) = \nu_{P_i}(W). \quad (4.110)$$

Hence

$$\nu_{P_i}(k_{j_1, j_2}^{(i)} \tau_i) = -j_1 - 1 \quad \text{for } j_1 \geq 0, j_2 \in [1, e_i] \text{ and } i \in [1, s]. \quad (4.111)$$

By (4.107) and (4.109), we have

$$\operatorname{div}(k_{j_1, j_2}^{(i)}) + \operatorname{div}(dt_{s+1}) + (j_1 + 1)P_i + a_{j_1}P_{s+1} \geq 0 \quad (4.112)$$

for  $j_1 \geq 0, j_2 \in [1, e_i], i \in [1, s]$  and some  $a_{j_1} \in \mathbb{Z}$ . According to (3.18) and (3.20), we get  $f_n \in \mathcal{L}((m + g - 1)P_{s+1})$ . Therefore

$$\nu_P(f_n k_{j_1, j_2}^{(i)} dt_{s+1}) \geq 0 \quad \text{and} \quad \operatorname{Res}_P(f_n k_{j_1, j_2}^{(i)} dt_{s+1}) = 0 \quad \text{for all } P \in \mathbb{P}_f \setminus \{P_i, P_{s+1}\}.$$

Applying the Residue Theorem, we derive

$$\operatorname{Res}_{P_i}(f_n k_{j_1, j_2}^{(i)} dt_{s+1}) = -\operatorname{Res}_{P_{s+1}}(f_n k_{j_1, j_2}^{(i)} dt_{s+1}) \quad (4.113)$$

for  $j_1 \geq 0, j_2 \in [1, e_i]$  and  $i \in [1, s]$ . Using (4.111), we get the following local expansion

$$\tau_i k_{j_1, j_2}^{(i)} := \sum_{r=-j_1}^{\infty} \kappa_{j_1, r}^{(i, j_2)} t_i^{r-1}, \quad \text{where all } \kappa_{j_1, r}^{(i, j_2)} \in \mathbb{F}_b \text{ and } \kappa_{j_1, j_1}^{(i, j_2)} \neq 0$$

for  $j_1 \geq 0, j_2 \in [1, e_i]$  and  $i \in [1, s]$ . By (3.23) and (4.113), we obtain

$$\begin{aligned} -\operatorname{Res}_{P_{s+1}, t_{s+1}}(f_n k_{j_1, j_2}^{(i)}) &= \operatorname{Res}_{P_i, t_i}(f_n \tau_i k_{j_1, j_2}^{(i)}) = \operatorname{Res}_{P_i, t_i} \left( \sum_{j=0}^{\infty} f_{n, j}^{(i)} t_i^j \sum_{r=-j_1}^{\infty} \kappa_{j_1, r}^{(i, j_2)} t_i^{r-1} \right) \\ &= \sum_{j=0}^{\infty} \sum_{r=-j_1}^0 \operatorname{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{n, j}^{(i)} \kappa_{j_1, r}^{(i, j_2)}) \delta_{j, -r} = \sum_{j=0}^{j_1} \operatorname{Tr}_{F_{P_i}/\mathbb{F}_b}(f_{n, j}^{(i)} \kappa_{j_1, -j}^{(i, j_2)}) = 0 \end{aligned} \quad (4.114)$$

for  $0 \leq j_1 \leq d_i - 1$ ,  $1 \leq j_2 \leq e_i$  and  $1 \leq i \leq s$ . Similarly to the proof of Lemma 14, we get from (4.114) the assertion of Lemma 18.  $\square$

**Lemma 19.** *Let  $s \geq 2$ ,  $d_0 = d + t$ ,  $\epsilon = \eta_1(2sd_0e)^{-1}$ ,  $\eta_1 = (1 + \deg((t_{s+1})_\infty))^{-1}$ ,*

$$\Lambda_1 = \left\{ \left( \left( \operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1, j_2}^{(i)}) \right)_{\substack{d_{i,1} \leq j_1 \leq d_{i,2}, 1 \leq i \leq s \\ 1 \leq j_2 \leq e_i}}, \bar{a}_{d_{s+1,1}}(n), \dots, \bar{a}_{d_{s+1,2}}(n) \right) | n \in [0, b^m] \right\}$$

with  $e = e_1 e_2 \cdots e_s$ ,  $e_{s+1} = 1$ ,  $d_{s+1,1} = t + (s-1)d_0[m\epsilon]e$ ,

$$d_{s+1,2} = t - 1 + sd_0[m\epsilon]e, \quad d_{i,1} = q_i, \quad d_{i,2} = d_0[m\epsilon]e/e_i - g - 1 \text{ for } i \in [1, s], \quad (4.115)$$

and  $m \geq |2g - 2 + 2(t + g - 2)(\eta_1^{-1} - 1)| + 2t + 2/\epsilon$ . Then

$$\Lambda_1 = \mathbb{F}_b^\chi \quad \text{with} \quad \chi = \sum_{i=1}^{s+1} (d_{i,2} - d_{i,1} + 1)e_i. \quad (4.116)$$

**Proof.** Suppose that (4.116) is not true. We get that there exists  $b_{j_1, j_2}^{(i)} \in \mathbb{F}_b$  ( $i, j_1, j_2 \geq 1$ ) such that

$$\sum_{i=1}^s \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b_{j_1, j_2}^{(i)}| + \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} |b_{j_1}^{(s+1)}| > 0 \quad (4.117)$$

and

$$\sum_{i=1}^s \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b_{j_1, j_2}^{(i)} \operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1, j_2}^{(i)}) + \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} b_{j_1}^{(s+1)} \bar{a}_{j_1}(n) = 0 \quad (4.118)$$

for all  $n \in [0, b^m]$ . From (3.18)-(3.20), we obtain the following local expansion

$$f_n = \dot{f}_n + \ddot{f}_n = \sum_{r \leq m+g-1} f_{n,r}^{(s+1)} t_{s+1}^{-r}, \quad \text{with} \quad \ddot{f}_n = \sum_{i=g}^{m-1} \bar{a}_i(n) v_i, \quad (4.119)$$

and  $\dot{f}_n = \sum_{i=0}^{g-1} \bar{a}_i(n) v_i$ , where  $n \in [0, b^m]$ . Let  $r \geq g$ .

Using (3.18)-(3.20) and (3.28), we derive that  $\nu_{P_{s+1}}(\dot{f}_n) \geq -2g + 1$ ,  $\nu_{P_{s+1}}(\dot{f}_n t_{s+1}^{r+g-1}) \geq 0$  and

$$f_{n,r+g}^{(s+1)} = \operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n t_{s+1}^{r+g-1}) = \operatorname{Res}_{P_{s+1}, t_{s+1}} (\ddot{f}_n t_{s+1}^{r+g-1}) = \operatorname{Res}_{P_{s+1}, t_{s+1}} \left( \sum_{i=g}^{m-1} \bar{a}_i(n) \right)$$

$$\times \sum_{j \leq i+g} v_{i,j} t_{s+1}^{-j+r+g-1}) = \sum_{i=g}^{m-1} \bar{a}_i(n) \sum_{j \leq i+g} v_{i,j} \delta_{j,r+g} = \sum_{m-1 \geq i \geq r} \bar{a}_i(n) v_{i,r+g} \text{ for } r \geq g.$$

Taking into account that  $v_{i,i+g} = 1$  and  $v_{i,r+g} = 0$  for  $i > r \geq g$  (see (3.29)), we get

$$f_{n,r+g}^{(s+1)} = \bar{a}_r(n) \quad \text{for } r \geq g \quad \text{and } n \in [0, b^m). \quad (4.120)$$

By (4.118), we have

$$\sum_{i=1}^s \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b_{j_1,j_2}^{(i)} \operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1,j_2}^{(i)}) + \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} b_{j_1}^{(s+1)} \operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n t_{s+1}^{j_1+g-1}) = 0$$

for all  $n \in [0, b^m)$ . Hence

$$\operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n \alpha) = 0 \quad \text{for all } n \in [0, b^m), \quad \text{where } \alpha = \alpha_1 + \alpha_2, \quad (4.121)$$

$$\alpha_1 = \sum_{i=1}^s \alpha_{1,i}, \quad \alpha_{1,i} = \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} b_{j_1,j_2}^{(i)} k_{j_1,j_2}^{(i)}, \quad \text{and } \alpha_2 = \sum_{j_1=d_{s+1,1}}^{d_{s+1,2}} b_{j_1}^{(s+1)} t_{s+1}^{j_1+g-1}.$$

According to (4.108), we get the following local expansion

$$k_{j_1,j_2}^{(i)} := \sum_{r=g+1}^{\infty} \chi_{j_1,r}^{(i,j_2)} t_{s+1}^{r-1}, \quad \text{where all } \chi_{j_1,r}^{(i,j_2)} \in \mathbb{F}_b,$$

and

$$\alpha = \sum_{r=g+1}^{\infty} \varphi_r t_{s+1}^{r-1} \quad \text{with } \varphi_r \in \mathbb{F}_b, \quad r \geq g+1. \quad (4.122)$$

Using (2.12) and (4.119)-(4.121), we have

$$\begin{aligned} \operatorname{Res}_{P_{s+1}, t_{s+1}} (f_n \alpha) &= \operatorname{Res}_{P_{s+1}, t_{s+1}} \left( \sum_{j \leq m+g-1} f_{n,j}^{(s+1)} t_{s+1}^{-j} \sum_{r=g+1}^{\infty} \varphi_r t_{s+1}^{r-1} \right) \\ &= \sum_{j \leq m+g-1} f_{n,j}^{(s+1)} \sum_{r=g+1}^{\infty} \varphi_r \delta_{j,r} = \sum_{j=g+1}^{m+g-1} f_{n,j}^{(s+1)} \varphi_j = \sum_{r=g+1}^{m+g-1} \bar{a}_r(n) \varphi_r = 0. \end{aligned}$$

for  $n \in [0, b^m)$ . Hence

$$\varphi_r = 0 \quad \text{for } g+1 \leq r \leq m+g-1.$$

By (4.122), we obtain

$$\nu_{P_{s+1}}(\alpha) \geq m + g - 1.$$

Applying (4.106), (4.107) and (4.121), we derive

$$\alpha \in \mathcal{L}(G_1), \text{ with } G_1 = W + \sum_{i=1}^s d_{i,2}P_i + (d_{s+1,2} + g - 1)(t_{s+1})_\infty - (m + g - 1)P_{s+1}.$$

From (4.115), we have

$$\begin{aligned} \deg(G_1) &= 2g - 2 + \sum_{i=1}^s d_{i,2}e_i + (d_{s+1,2} + g - 1)\deg((t_{s+1})_\infty) - (m + g - 1) \\ &\leq 2g - 2 + sd_0e[m\epsilon] + (t - 1 + sd_0e[m\epsilon] + g - 1)(\eta_1^{-1} - 1) - (m + g - 1) \\ &\leq g - 1 + (t + g - 2)(\eta_1^{-1} - 1) + sd_0em\epsilon\eta_1^{-1} - m = g - 1 + (t + g - 2)(\eta_1^{-1} - 1) - m/2 < 0 \end{aligned}$$

for  $m > 2g - 2 + 2(t + g - 2)(\eta_1^{-1} - 1)$ . Hence  $\alpha = 0$ .

Suppose that  $\sum_{i=1}^s \sum_{j_1=d_{i,1}}^{d_{i,2}} \sum_{j_2=1}^{e_i} |b_{j_1,j_2}^{(i)}| = 0$ . Then  $\alpha_2 = 0$ . From (4.121), we derive  $b_{j_1}^{(s+1)} = 0$  for all  $j_1 \in [d_{s+1,1}, d_{s+1,2}]$ . According to (4.117), we have a contradiction. Hence there exists  $h \in [1, s]$  with

$$\sum_{j_1=d_{h,1}}^{d_{h,2}} \sum_{j_2=1}^{e_h} |b_{j_1,j_2}^{(h)}| > 0. \quad (4.123)$$

Let  $h > 1$ . By (3.27) and (4.121), we get  $\nu_{P_h}(t_{s+1}) \geq 0$  and  $\nu_{P_h}(\alpha_2) \geq 0$ . Applying (2.3) and (2.4), we derive  $\nu_{P_h}(W) = \nu_{P_h}(dt_{s+1}) = \nu_{P_h}(dt_{s+1}/dt_h) \geq 0$ .

By (4.112), we have  $\nu_{P_h}(\alpha_{1,j}) \geq -\nu_{P_h}(W)$  for  $1 \leq j \leq s, j \neq h$ . Taking into account that  $\alpha_{1,h} = -\sum_{1 \leq j \leq s, j \neq h} \alpha_{1,j} - \alpha_2$ , we get  $\nu_{P_h}(\alpha_{1,h}) \geq -\nu_{P_h}(W)$ .

Using (4.110) and (4.111), we obtain  $\nu_{P_h}(k_{j_1,j_2}^{(h)}) = -j_1 - 1 - \nu_{P_h}(W)$ . Bearing in mind (4.123) and that  $\{u_1^{(i)}, \dots, u_{z_i}^{(i)}, k_{q_i,1}^{(i)}, \dots, k_{q_i,e_1}^{(i)}, \dots, k_{n,1}^{(i)}, \dots, k_{n,e_1}^{(i)}\}$  is a basis of  $\mathcal{L}(G_i + (n - q_i + 1)P_i)$ , we get

$$\alpha_{1,h} \in \mathcal{L}(G_i + (d_{i,2} - q_i + 1)P_i) \setminus \mathcal{L}(G_i + (d_{i,1} - q_i)P_i).$$

From (4.115) and (4.121), we derive  $\nu_{P_h}(\alpha_{1,h}) \leq -\nu_{P_h}(W) - 1$ . We have a contradiction.

Now let  $h = 1$  and (4.123) is not true for  $h \in [2, s]$ . Hence  $\alpha_{1,1} = -\alpha_2$  and  $\nu_{P_{s+1}}(\alpha_{1,1}) \geq d_{s+1,1} + g - 1$ . By (4.106), (4.107) and (4.121), we have

$$\alpha_{1,1} \in \mathcal{L}(\dot{G}) \quad \text{with} \quad \dot{G} = W + (d_{1,2} + 1)P_1 - (d_{s+1,1} + g - 1)P_{s+1}.$$

From (4.115), we get

$$\deg(\dot{G}) = 2g - 2 + d_0e[m\epsilon] - ge_1 - (s - 1)d_0e[m\epsilon] - g + 1 \leq 2g - 2 - 2g + 1 < 0.$$

Hence  $\alpha_{1,1} = 0$ . Therefore (4.123) is not true for  $h = 1$ . We have a contradiction. Thus assertion (4.117) is not true, and Lemma 19 follows.  $\square$

#### End of the proof of Theorem 4.

Let  $\tilde{d}_{i,2} = d_{i,2} + g = d_0[m\epsilon]e/e_i - 1$  ( $1 \leq i \leq s$ ),

$$\Lambda'_1 = \left\{ \left( \left( \text{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1, j_2}^{(i)}) \right)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s}, \bar{a}_{d_{s+1,1}}(n), \dots, \bar{a}_{d_{s+1,2}}(n) \right) \middle| n \in [0, b^m) \right\}$$

and

$$\Lambda_2 = \left\{ (\bar{a}_{d_{s+1,1}}(n), \dots, \bar{a}_{d_{s+1,2}}(n)) \mid \text{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1, j_2}^{(i)}) = 0 \right. \\ \left. \text{for } 0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s, n \in [0, b^m) \right\}.$$

By (4.97) and Lemma 19, we have  $\dim_{\mathbb{F}_b}(\Lambda'_1) \geq \dim_{\mathbb{F}_b}(\Lambda_1)$  and

$$\begin{aligned} \dim_{\mathbb{F}_b}(\Lambda_2) &= \dim_{\mathbb{F}_b}(\Lambda'_1) - \dim_{\mathbb{F}_b} \left( \left\{ \left( \text{Res}_{P_{s+1}, t_{s+1}} (f_n k_{j_1, j_2}^{(i)}) \right)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s} \middle| n \in [0, b^m) \right\} \right) \\ &\geq \dim_{\mathbb{F}_b}(\Lambda_1) - \sum_{i=1}^s (\tilde{d}_{i,2} + 1)e_i \geq d_{s+1,2} - d_{s+1,1} + 1 - \sum_{i=1}^s (q_i + g)e_i. \end{aligned} \quad (4.124)$$

Using Lemma 18, we get  $\Lambda_3 \supseteq \Lambda_2$  and  $\dim_{\mathbb{F}_b}(\Lambda_3) \geq \dim_{\mathbb{F}_b}(\Lambda_2)$ , where

$$\Lambda_3 = \left\{ (\bar{a}_{d_{s+1,1}}(n), \dots, \bar{a}_{d_{s+1,2}}(n)) \mid \text{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n, j_1}^{(i)}) = 0 \right. \\ \left. \text{for } 0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s, n \in [0, b^m) \right\}.$$

Taking into account that  $(\mathbf{x}_n)_{0 \leq n < b^m}$  is a  $(t, m, s)$  net in base  $b$ , we get from (3.24) and (3.25) that

$$\left\{ \left( f_{n, j_1}^{(i)} \right)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq i \leq s} \middle| n \in [0, b^m) \right\} = \prod_{i=1}^s F_{P_i}^{\tilde{d}_{i,2}+1}.$$

Bearing in mind that  $\{\beta_1^{(i)}, \dots, \beta_{e_i}^{(i)}\}$  is a basis of  $F_{P_i}/\mathbb{F}_b$  (see Lemma 18), we obtain

$$\Lambda_4 = \left\{ \left( \text{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) \right)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s} \mid n \in [0, b^m] \right\} = \mathbb{F}_b^{sd_0 e [m\epsilon]}.$$

Let

$$\Lambda_5 = \left\{ \left( \text{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) \right)_{0 \leq j_1 \leq \tilde{d}_{i,2}, 1 \leq j_2 \leq e_i, 1 \leq i \leq s}, (\bar{a}_j(n))_{d_{s+1,1} \leq j \leq d_{s+1,2}} \mid n \in [0, b^m] \right\}.$$

By (4.124), (4.97) and (4.106), we have

$$\dim_{\mathbb{F}_b}(\Lambda_5) = \dim_{\mathbb{F}_b}(\Lambda_3) + \dim_{\mathbb{F}_b}(\Lambda_4) \geq d_{s+1,2} - d_{s+1,1} + 1 + sd_0 e \dot{m} - r$$

with  $r = (g+1)(e_0 + s)$ ,  $e = e_1 e_2 \dots e_s$  and  $\dot{m} = [m\epsilon]$ .

Let  $\dot{m}_1 = d_0 e \dot{m}$ ,  $\epsilon = \eta_1 (2sd_0 e)^{-1}$ ,  $\ddot{m}_i = 0$ ,  $1 \leq i \leq s$ , and  $\ddot{m}_{s+1} = d_{s+1,1} + g$ ,  $d_{s+1,1} = t + (s-1)d_0[m\epsilon]e$ ,  $d_{s+1,2} = t - 1 + sd_0[m\epsilon]e = d_{s+1,1} + \dot{m}_1 - 1$  (see (4.115)),  $\tilde{d}_{i,2} = d_0[m\epsilon]e/e_i - 1 = \tilde{d}_{i,2} + g = \dot{m}_1/e_i - 1$  ( $i \in [1, s]$ ),

$$\dot{\theta}_{n,j_1 e_s + j_2}^{(i)} := \text{Tr}_{F_{P_i}/\mathbb{F}_b}(\beta_{j_2}^{(i)} f_{n,j_1}^{(i)}) \quad \text{and} \quad \dot{\theta}_{n,j+1}^{(s+1)} := f_{n,j}^{(s+1)} = \bar{a}_{j-g}(n) \quad (\text{see (4.120)})$$

for  $0 \leq j_1 \leq \tilde{d}_{i,2}$ ,  $1 \leq j_2 \leq e_i$ ,  $1 \leq i \leq s$ ,  $2g \leq j$ , and let

$$\Lambda_6 = \left\{ \left( \left( \dot{\theta}_{\ddot{m}_i + d_0 e \dot{j}_i + \ddot{j}_i}^{(i)} \right)_{0 \leq \dot{j}_i < \dot{m}, 1 \leq \ddot{j}_i \leq d_0 e, 1 \leq i \leq s+1} \mid n \in [0, b^m] \right) \right\}.$$

It is easy to verify that  $\Lambda_6 = \Lambda_5$  and  $\dim_{\mathbb{F}_b}(\Lambda_6) = (s+1)\dot{m}_1 - \dot{r}$  with  $0 \leq \dot{r} \leq r = (g+1)(e_0 + s)$ .

Let  $m \geq |2g - 2 + 2(t + g - 2)(\eta_1^{-1} - 1)| + 2t + 2/\epsilon$ . Applying Lemma 2, with  $\dot{s} = s+1$ , we get that there exists  $B_i \subset \{0, \dots, \dot{m} - 1\}$ ,  $1 \leq i \leq s+1$  such that

$$\Lambda_7 = \mathbb{F}_b^{(s+1)\dot{m}_1 - d_0 e B}, \quad \text{where} \quad B = \#B_1 + \dots + \#B_{s+1} \leq (g+1)(e_0 + s),$$

and

$$\Lambda_7 = \left\{ \left( \dot{\theta}_{\ddot{m}_i + d_0 e \dot{j}_i + \ddot{j}_i}^{(i)} \mid \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, d_0 e], i \in [1, s+1] \right) \mid n \in [0, b^m] \right\}$$

with  $\bar{B}_i = \{0, \dots, \dot{m} - 1\} \setminus B_i$ . Hence

$$\left\{ \left( f_{n, \ddot{m}_i + \dot{j}_i d_0 e / e_i + \ddot{j}_i - 1}^{(i)} \mid \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, \frac{d_0 e}{e_i}], i \in [1, s+1] \right) \mid n \in [0, b^m] \right\} = \prod_{i=1}^s F_{P_i}^{\chi_i} \mathbb{F}_b^{\chi_{s+1}}$$



with  $e_{s+1} = 1$ ,  $\chi_i = d_0 e(\dot{m} - \#B_i)/e_i$ ,  $1 \leq i \leq s+1$ .

Taking into account that  $\sigma_{P_i} : F_{P_i} \rightarrow Z_{b^{e_i}}$  is a bijection (see (3.21)), we obtain

$$\left\{ \left( \sigma_{P_i}(f_{n, \ddot{m}_i + \dot{j}_i d_0 e / e_i + \ddot{j}_i - 1}^{(i)}) \mid \dot{j}_i \in \bar{B}_i, \ddot{j}_i \in [1, \frac{d_0 e}{e_i}], i \in [1, s] \right), \right.$$

$$\left. (a_{\ddot{m}_{s+1} + \dot{j}_{s+1} d_0 e + \ddot{j}_{s+1} - 1 - g}(n) \mid \dot{j}_{s+1} \in \bar{B}_{s+1}, \ddot{j}_{s+1} \in [1, d_0 e]) \mid n \in [0, b^m] \right\} = Z_b^{(s+1)\dot{m}_1 - d_0 e B}.$$

Let  $\tilde{B}_i = \bar{B}_i$ ,  $1 \leq i \leq s$ , and let  $\tilde{B}_{s+1} = \{\dot{m} - j - 1 \mid j \in \bar{B}_{s+1}\}$ . From (4.103), we derive

$$\left\{ \left( x_{n, \ddot{m}_i + \dot{j}_i d_0 e + \ddot{j}_i - 1}^{(i)} \mid \dot{j}_i \in \tilde{B}_i, \ddot{j}_i \in [1, d_0 e], i \in [1, s+1] \right) \mid n \in [0, b^m] \right\} = Z_b^{(s+1)\dot{m}_1 - d_0 e B},$$

where  $x_n^{(s+1)} = \sum_{j=1}^m x_{n,j}^{(s+1)} b^{-j} := n/b^m$ , and  $x_{n,j}^{(s+1)} = a_{m-j-1}(n)$  ( $1 \leq j \leq m$ ),  $\ddot{m}_i = \ddot{m}_i = 0$  for  $1 \leq i \leq s$  and  $\ddot{m}_{s+1} = m - t - s\dot{m}_1 = m - 1 - (\ddot{m}_{s+1} + \dot{m}_1 - 1 - g)$ .

By Lemma 17 and Theorem K, we obtain that  $(\mathbf{x}_n)_{n \geq 0}$  is a  $d$ -admissible  $(t, s)$  sequence with  $\mathbf{x}_n = (x_n^{(1)}, \dots, x_n^{(s)})$ ,  $d = g + e_0$  and  $t = g + e_0 - s$ .

Now applying Corollary 1 with  $\dot{s} = s+1$ ,  $\tilde{r} = 0$ ,  $\tilde{m} = m$  and  $\hat{e} = e = e_1 \dots e_{s+1}$ , we derive

$$\min_{0 \leq Q < b^m} \min_{\mathbf{w} \in E_m^s} b^m D^*((\mathbf{x}_n \oplus \mathbf{w}, n \oplus Q/b^m)_{0 \leq n < b^m}) \geq 2^{-2} b^{-d} K_{d,t,s+1}^{-s} \eta_1^s m^s,$$

with  $m \geq 2^{2s+3} b^{d+t+s+1} (d+t)^{s+1} s^{2s} e(g+1)(e_0+s) \eta_1^{-s}$ , and  $\eta_1 = (1 + \deg((t_{s+1})_\infty))^{-1}$ . Using Lemma B, we get the first assertion in Theorem 4.

Consider the second assertion in Theorem 4.

By (3.23)-(3.25), we get that the net  $(\mathbf{x}_n)_{0 \leq n < b^m}$  is constructed similarly to the construction of the Niederreiter-Özbudak net (see (4.61)-(4.69) and (3.15)). The difference is that in the construction of Section 2.3 the map  $\sigma_i : F_{P_i} \rightarrow \mathbb{F}_b^{e_i}$  is linear, while in the construction of Section 2.4 this map may be nonlinear.

It is easy to verify that this does not affect the proof of bound (3.31) and Theorem 4 follows.  $\square$

## 4.5 Niederreiter-Xing sequence. Sketch of the proof of Theorem 5

First we will prove that

$$\dot{C}_m = \mathcal{M}_m^\perp(P_1, \dots, P_s; G_m) \quad \text{for } m \geq g+1. \quad (4.125)$$

By (2.20) and (3.34), we get

$$\dot{\mathcal{C}}_m = \left\{ \left( \sum_{r=0}^{m-1} \dot{c}_{j,r}^{(i)} \bar{a}_r(n) \right)_{0 \leq j \leq m-1, 1 \leq i \leq s} \mid 0 \leq n < b^m \right\}.$$

Using (4.58) with  $\tilde{G} = (g-1)P_{s+1}$ , we derive  $G_m^\perp = L_m$ , where  $L_m = \mathcal{L}((m-g+1)P_{s+1} + W)$ . From (3.33), we have

$$\{f^\perp \mid f^\perp \in L_m\} = \{\dot{f}_n := \sum_{r=0}^{m-1} a_r(n) \dot{v}_r \mid n \in [0, b^m)\}.$$

Applying (3.34), we obtain

$$\dot{f}_n \tau_i = \sum_{j=0}^{\infty} \dot{f}_{n,j}^{(i)} t_i^j, \quad \text{where} \quad \dot{f}_{n,j}^{(i)} = \sum_{r=0}^{m-1} \dot{c}_{j,r}^{(i)} \bar{a}_r(n) \in \mathbb{F}_b, \quad i \in [1, s], \quad j \geq 0.$$

Therefore

$$\dot{\mathcal{C}}_m = \{(\dot{f}_{n,j}^{(i)})_{0 \leq j \leq m-1, 1 \leq i \leq s} \mid 0 \leq n < b^m\}. \quad (4.126)$$

We use notations (4.59)-(4.69) with the following modifications. In (4.61) we take the field  $\mathbb{F}_b$  instead of  $F_{P_i}$ , and in (4.62) we consider the map  $\vartheta_i^\perp$  as the identical map ( $1 \leq i \leq s$ ). By (4.63), we have  $\dot{\theta}_{i,j}^\perp(f_n) = \dot{f}_{n,j-1}^{(i)}$  for  $1 \leq j \leq m$ , and  $\dot{\theta}_i^\perp(\dot{f}_n) = (\dot{f}_{n,0}^{(i)}, \dots, \dot{f}_{n,m-1}^{(i)})$ ,  $1 \leq i \leq s$ . According to (4.69) and (4.126) we get

$$\begin{aligned} \Xi_m = \dot{\Xi}_m &= \{\dot{\theta}^\perp(f^\perp) \mid f^\perp \in \mathcal{L}(G_m^\perp)\} = \{\dot{\theta}^\perp(\dot{f}_n) \mid n \in [0, b^m)\} \\ &= \{(\dot{\theta}_1^\perp(\dot{f}_n), \dots, \dot{\theta}_s^\perp(\dot{f}_n)) \mid n \in [0, b^m)\} = \{(\dot{f}_{n,j}^{(i)})_{0 \leq j \leq m-1, 1 \leq i \leq s} \mid 0 \leq n < b^m\} = \dot{\mathcal{C}}_m. \end{aligned}$$

Now applying (3.13), (3.32) and Lemma 12, we obtain (4.125). By [DiPi, ref. 8.9], we have

$$\delta_m(\mathcal{M}_m) = \delta_m(\mathcal{M}_m(P_1, \dots, P_s; G_m)) \geq m - g + 1 \quad \text{for} \quad m \geq g + 1.$$

Taking into account Proposition B, we get that  $\mathbf{x}_n(\dot{C})_{n \geq 0}$  is a digital  $(\mathbf{T}, s)$  sequence with  $T(m) = g$  for  $m \geq g + 1$ .

Now the  $d$ -admissible property follow from Lemma 16. In order to complete the proof of Theorem 5, we use Theorem 3 and Theorem 4.  $\square$

## 4.6 General $d$ -admissible $(t, s)$ sequences. Proof of Theorem 6.

First we will prove Lemma 20. We need the following notations:

Let  $\tilde{C}^{(1)}, \dots, \tilde{C}^{(\dot{s})}$  are  $m \times m$  generating matrices of a digital  $(t, m, \dot{s})$  net  $(\tilde{\mathbf{x}}_n)_{n=0}^{b^m-1}$  in base  $b$ ,  $\tilde{x}_n^{(\dot{s})} \neq \tilde{x}_k^{(\dot{s})}$  for  $n \neq k$ ,  $\tilde{C}^{(i)} = (\tilde{c}_{r,j}^{(i)})_{1 \leq r,j \leq m}$ ,  $\tilde{\mathbf{c}}_j^{(i)} = (\tilde{c}_{1,j}^{(i)}, \dots, \tilde{c}_{m,j}^{(i)}) \in \mathbb{F}_b^m$ ,  $i \in [1, \dot{s}]$ ,  $\tilde{\mathbf{c}}_j = (\tilde{\mathbf{c}}_j^{(1)}, \dots, \tilde{\mathbf{c}}_j^{(\dot{s})}) \in \mathbb{F}_b^{m\dot{s}}$  ( $1 \leq j \leq m$ ). Let  $\phi : Z_b \mapsto \mathbb{F}_b$  be a bijection with  $\phi(0) = \bar{0}$ , and let  $n = \sum_{j=1}^m a_j(n)b^{j-1}$ ,  $\mathbf{n} = (\bar{a}_1(n), \dots, \bar{a}_m(n)) \in \mathbb{F}_b^m$ ,  $\bar{a}_j(n) = \phi(a_j(n))$ ,  $\tilde{\mathbf{y}}_n = (\tilde{\mathbf{y}}_n^{(1)}, \dots, \tilde{\mathbf{y}}_n^{(\dot{s})}) \in \mathbb{F}_b^{m\dot{s}}$ ,  $\tilde{\mathbf{y}}_n^{(i)} = (\tilde{y}_{n,1}^{(i)}, \dots, \tilde{y}_{n,m}^{(i)}) \in \mathbb{F}_b^m$ ,

$$\tilde{\mathbf{x}}_n = (\tilde{x}_n^{(1)}, \dots, \tilde{x}_n^{(\dot{s})}), \quad \tilde{x}_n^{(i)} = \sum_{j=1}^m \phi^{-1}(\tilde{y}_{n,j}^{(i)})/b^j \quad \text{for } 1 \leq i \leq \dot{s}, \quad (4.127)$$

$$\tilde{\mathbf{y}}_n^{(i)} = \mathbf{n}(\tilde{\mathbf{c}}_1^{(i)}, \dots, \tilde{\mathbf{c}}_m^{(i)})^\top := \sum_{j=1}^m \bar{a}_j(n)\tilde{\mathbf{c}}_j^{(i)} = \mathbf{n}\tilde{C}^{(i)\top} \quad \text{for } 1 \leq i \leq \dot{s}. \quad (4.128)$$

Hence

$$\tilde{\mathbf{y}}_n = \sum_{j=1}^m \bar{a}_j(n)\tilde{\mathbf{c}}_j, \quad \text{for } 0 \leq n < b^m.$$

We put

$$\tilde{\Phi}_m = \{\tilde{\mathbf{x}}_n | n \in [0, b^m)\}, \quad \tilde{\Psi}_m = \{\tilde{\mathbf{y}}_n | n \in [0, b^m)\}, \quad \tilde{Y}_m = \{\tilde{\mathbf{y}}_n^{(\dot{s})} | n \in [0, b^m)\}.$$

We see that  $\tilde{\Psi}_m$  is a vector space over  $\mathbb{F}_b$ , with  $\dim(\tilde{\Psi}_m) \leq m$ . Taking into account that  $\tilde{x}_n^{(\dot{s})} \neq \tilde{x}_k^{(\dot{s})}$  for  $n \neq k$ , we obtain  $\dim(\tilde{\Psi}_m) = m$ ,  $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_m$  is the basis of  $\tilde{\Psi}_m$  and  $\tilde{Y}_m = \mathbb{F}_b^m$ .

Let  $d \geq 1$ ,  $d_0 = d + t$ ,  $m \geq 4d_0(s+1)$ ,  $\dot{m} = [(m-t)/(2d_0(\dot{s}-1))]$ ,

$$d_1^{(\dot{s})} = m - t + 1 - (\dot{s}-1)d_0\dot{m} \quad \text{and} \quad d_2^{(\dot{s})} = m - t - (\dot{s}-2)d_0\dot{m}. \quad (4.129)$$

Bearing in mind that  $\tilde{\Phi}_m$  is a  $(t, m, \dot{s})$  net, we get that for each  $j \in [1, (\dot{s}-1)d_0\dot{m}]$  with  $j = (j_1-1)(\dot{s}-1) + j_2$ ,  $j_1 \in [1, d_0\dot{m}]$  and  $j_2 \in [1, \dot{s}-1]$  there exists  $n(j) \in [0, b^m)$  such that

$$\tilde{x}_{n(j),r_1}^{(\dot{s})} = \delta_{(j_1-1)(\dot{s}-1)+j_2,r_1} \quad \text{and} \quad \tilde{x}_{n(j),r_2}^{(i)} = \delta_{i,j_2}\delta_{j_1,r_2} \quad (4.130)$$

for all  $r_1 \in [1, (\dot{s}-1)d_0\dot{m}]$ ,  $r_2 \in [1, d_0\dot{m}]$ ,  $i \in [1, \dot{s}-1]$ .

Taking into account that  $Y_m = \mathbb{F}_b^m$ , we derive that there exists  $n(j) \in [0, b^m)$  with

$$\tilde{y}_{n(j),r}^{(s)} = \delta_{j,r} \quad \text{for} \quad (\dot{s} - 1)d_0\dot{m} + 1 \leq j \leq m, \quad 1 \leq r \leq m. \quad (4.131)$$

We take a basis  $\dot{\mathbf{f}}_1, \dots, \dot{\mathbf{f}}_m$  of  $\tilde{\Psi}_m$  in the following way:

Let  $\dot{\mathbf{f}}_j = (\dot{\mathbf{f}}_j^{(1)}, \dots, \dot{\mathbf{f}}_j^{(\dot{s})}) \in \mathbb{F}_b^{m\dot{s}}$  with  $\dot{\mathbf{f}}_j^{(i)} = (\dot{\mathbf{f}}_{1,j}^{(i)}, \dots, \dot{\mathbf{f}}_{m,j}^{(i)}) \in \mathbb{F}_b^m$ ,  $i \in [1, \dot{s}]$ ,  $j \in [1, m]$ . For  $j \in [1, m]$ , we put  $\dot{\mathbf{f}}_j := \tilde{\mathbf{y}}_{n(j)}$ . We have from (4.130) and (4.131) that

$$\dot{\mathbf{f}}_{(j_1-1)(\dot{s}-1)+j_2,r_1}^{(\dot{s})} = \delta_{(j_1-1)(\dot{s}-1)+j_2,r_1} \quad \text{and} \quad \dot{\mathbf{f}}_{(j_1-1)(\dot{s}-1)+j_2,r_2}^{(i)} = \delta_{i,j_2} \delta_{j_1,r_2}$$

for  $r_1 \in [1, (\dot{s} - 1)d_0\dot{m}]$ ,  $r_2 \in [1, d_0\dot{m}]$ ,  $i \in [1, \dot{s} - 1]$ ,  $j_1 \in [1, d_0\dot{m}]$ ,  $j_2 \in [1, \dot{s} - 1]$  and

$$\dot{\mathbf{f}}_{j,r}^{(\dot{s})} = \delta_{j,r} \quad \text{for} \quad (\dot{s} - 1)d_0\dot{m} + 1 \leq j \leq m, \quad 1 \leq r \leq m. \quad (4.132)$$

It is easy to see that the vectors  $\dot{\mathbf{f}}_1, \dots, \dot{\mathbf{f}}_m \in \tilde{\Psi}_m$  are linearly independent over  $\mathbb{F}_b$ . Thus  $\dot{\mathbf{f}}_1, \dots, \dot{\mathbf{f}}_m$  is a basis of  $\tilde{\Psi}_m$ .

Let

$$\dot{\mathbf{y}}_n^{(i)} = (\dot{y}_{n,1}^{(i)}, \dots, \dot{y}_{n,m}^{(i)}) := \mathbf{n}(\dot{\mathbf{f}}_1^{(i)}, \dots, \dot{\mathbf{f}}_m^{(i)}) = \sum_{j=1}^m \bar{a}_j(n) \dot{\mathbf{f}}_j^{(i)} = \mathbf{n} \dot{\mathcal{F}}^{(i)\top}, \quad (4.133)$$

where  $\dot{\mathcal{F}}^{(i)} = (\dot{\mathbf{f}}_{r,j}^{(i)})_{1 \leq r,j \leq m}$  for  $1 \leq i \leq \dot{s}$ . Hence

$$\dot{\mathbf{y}}_n := (\dot{\mathbf{y}}_n^{(1)}, \dots, \dot{\mathbf{y}}_n^{(\dot{s})}) = \sum_{j=1}^m \bar{a}_j(n) \dot{\mathbf{f}}_j \quad \text{for} \quad 0 \leq n < b^m.$$

We put

$$\dot{\Psi}_m = \{\dot{\mathbf{y}}_n \mid 0 \leq n < b^m\}.$$

It is easy to see that  $\dot{\Psi}_m = \tilde{\Psi}_m$ .

For  $\ddot{\mathbf{f}}_j = (\ddot{\mathbf{f}}_j^{(1)}, \dots, \ddot{\mathbf{f}}_j^{(\dot{s})})$  with  $\ddot{\mathbf{f}}_j^{(i)} = (\ddot{\mathbf{f}}_{1,j}^{(i)}, \dots, \ddot{\mathbf{f}}_{m,j}^{(i)})$ , we define

$$\ddot{\mathbf{f}}_j = \dot{\mathbf{f}}_j \quad \text{for} \quad j \in [(\dot{s} - 1)d_0\dot{m} + 1, m] \quad \text{and} \quad \ddot{\mathbf{f}}_j^{(i)} = \dot{\mathbf{f}}_j^{(i)} \quad \text{for} \quad i \in [1, \dot{s} - 1], \quad j \in [1, m],$$

$$\ddot{\mathbf{f}}_{j,r}^{(\dot{s})} = \bar{0} \quad \text{for} \quad j \in [1, (\dot{s} - 1)d_0\dot{m}], \quad r \in [d_1^{(\dot{s})}, d_2^{(\dot{s})}], \quad \text{and} \quad \ddot{\mathbf{f}}_{j,r}^{(\dot{s})} = \dot{\mathbf{f}}_{j,r}^{(\dot{s})} \quad (4.134)$$

for  $j \in [1, (\dot{s} - 1)d_0\dot{m}]$  and  $r \in [1, m] \setminus [d_1^{(\dot{s})}, d_2^{(\dot{s})}]$ . Let

$$\ddot{\mathbf{y}}_n^{(i)} = (\ddot{y}_{n,1}^{(i)}, \dots, \ddot{y}_{n,m}^{(i)}) := \mathbf{n}(\ddot{\mathbf{f}}_1^{(i)}, \dots, \ddot{\mathbf{f}}_m^{(i)}) = \sum_{j=1}^m \bar{a}_j(n) \ddot{\mathbf{f}}_j^{(i)} = \mathbf{n} \ddot{\mathcal{F}}^{(i)\top}, \quad (4.135)$$

where  $\ddot{\mathcal{F}}^{(i)} = (\ddot{\mathbf{f}}_{r,j}^{(i)})_{1 \leq r, j \leq m}$  for  $1 \leq i \leq \dot{s}$ . Hence

$$\ddot{\mathbf{y}}_n := (\ddot{y}_n^{(1)}, \dots, \ddot{y}_n^{(\dot{s})}) = \sum_{j=1}^m \bar{a}_j(n) \ddot{\mathbf{f}}_j \quad \text{for } 0 \leq n < b^m. \quad (4.136)$$

We put

$$\ddot{\Psi}_m = \{\ddot{\mathbf{y}}_n \mid 0 \leq n < b^m\} \quad \text{and} \quad \ddot{Y}_m = \{\ddot{\mathbf{y}}_n^{(\dot{s})} \mid n \in [0, b^m]\}. \quad (4.137)$$

Now let  $\dot{\mathbf{x}}_n = (\dot{x}_n^{(1)}, \dots, \dot{x}_n^{(\dot{s})})$  and  $\ddot{\mathbf{x}}_n = (\ddot{x}_n^{(1)}, \dots, \ddot{x}_n^{(\dot{s})})$ , where

$$\dot{x}_n^{(i)} = \sum_{j=1}^m \phi^{-1}(\dot{y}_{n,j}^{(i)})/b^j, \quad \text{and} \quad \ddot{x}_n^{(i)} = \sum_{j=1}^m \phi^{-1}(\ddot{y}_{n,j}^{(i)})/b^j$$

for  $1 \leq i \leq \dot{s}$ . We have

$$\ddot{\Phi}_m = \{\ddot{\mathbf{x}}_n \mid 0 \leq n < b^m\} = \{\dot{\mathbf{x}}_n \mid 0 \leq n < b^m\} \quad \text{and} \quad \ddot{Y}_m = \mathbb{F}_b^m. \quad (4.138)$$

Bearing in mind that  $\dot{\mathbf{f}}_1, \dots, \dot{\mathbf{f}}_m$  and  $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_m$  are two basis of the vector space  $\ddot{\Psi}_m$ , we get that there exists a nonsingular matrix  $B = (b_{j,r})_{1 \leq j, r \leq m}$  with  $b_{j,r} \in \mathbb{F}_b$  such that  $(\dot{\mathbf{f}}_1, \dots, \dot{\mathbf{f}}_m)^\top = B(\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_m)^\top$ . Hence

$$\dot{\mathbf{f}}_k = \sum_{r=1}^m b_{k,r} \tilde{\mathbf{c}}_r, \quad \text{and} \quad \dot{\mathbf{f}}_{k,j}^{(i)} = \sum_{r=1}^m b_{k,r} \tilde{c}_{r,j}^{(i)},$$

for  $1 \leq k, j \leq m, 1 \leq i \leq \dot{s}$ . Therefore

$$(\dot{\mathbf{f}}_1^{(i)}, \dots, \dot{\mathbf{f}}_m^{(i)})^\top = B(\tilde{\mathbf{c}}_1^{(i)}, \dots, \tilde{\mathbf{c}}_m^{(i)})^\top \quad \text{and} \quad \tilde{C}^{(i)} = \dot{\mathcal{F}}^{(i)} B^{-1\top} \quad \text{for } i \in [1, \dot{s}]. \quad (4.139)$$

Let  $n' \in [0, b^m]$ ,  $\mathbf{n}' = (\bar{a}_1(n'), \dots, \bar{a}_m(n'))$ , and let  $\mathbf{n}' = \mathbf{n} B^{-1}$ .

Using (4.128) and (4.133), we get

$$\dot{\mathbf{y}}_{n'}^{(i)} = \mathbf{n}' \dot{\mathcal{F}}^{(i)\top} = \mathbf{n}' (\dot{\mathbf{f}}_1^{(i)}, \dots, \dot{\mathbf{f}}_m^{(i)})^\top = \mathbf{n} B^{-1} B(\tilde{\mathbf{c}}_1^{(i)}, \dots, \tilde{\mathbf{c}}_m^{(i)})^\top$$

$$= \mathbf{n}(\tilde{\mathbf{c}}_1^{(i)}, \dots, \tilde{\mathbf{c}}_m^{(i)})^\top = \mathbf{n}\tilde{C}^{(i)\top} = \tilde{\mathbf{y}}_n^{(i)}, \quad \text{for } 1 \leq i \leq \dot{s} \quad \text{and} \quad 0 \leq n < b^m.$$

Let  $\check{C}^{(i)} = (\check{c}_{r,j}^{(i)})_{1 \leq r,j \leq m} := \ddot{\mathcal{F}}^{(i)} B^{-1\top}$ ,  $1 \leq i \leq \dot{s}$ ,  $\check{\mathbf{c}}_j^{(i)} = (\check{c}_{1,j}^{(i)}, \dots, \check{c}_{m,j}^{(i)})$ ,  $1 \leq i \leq \dot{s}$ ,  $1 \leq j \leq m$  and let  $\check{\mathbf{y}}_n := \check{\mathbf{y}}_{n'}$ ,  $\check{\mathbf{x}}_n := \check{\mathbf{x}}_{n'}$  for  $\mathbf{n}' = \mathbf{n}B^{-1}$ . We have

$$\check{\mathbf{y}}_n^{(i)} = \check{\mathbf{y}}_{n'}^{(i)} = \mathbf{n}' \ddot{\mathcal{F}}^{(i)\top} = \mathbf{n}B^{-1} \ddot{\mathcal{F}}^{(i)\top} = \mathbf{n}\check{C}^{(i)\top} \quad \text{for } 1 \leq i \leq \dot{s}, \quad 0 \leq n < b^m. \quad (4.140)$$

Hence,  $\check{C}^{(1)}, \dots, \check{C}^{(\dot{s})}$  are generating matrices of the net  $(\check{\mathbf{x}}_n)_{0 \leq n < b^m}$ . According to (4.134) and (4.139), we obtain  $\check{\mathcal{F}}^{(i)} = \dot{\mathcal{F}}^{(i)}$ ,

$$\check{C}^{(i)} = \tilde{C}^{(i)} \quad \text{for } 1 \leq i \leq \dot{s} - 1, \quad \text{and} \quad \check{C}^{(\dot{s})} - \tilde{C}^{(\dot{s})} = (\ddot{\mathcal{F}}^{(\dot{s})} - \dot{\mathcal{F}}^{(\dot{s})})B^{-1\top}. \quad (4.141)$$

Let  $(B^{-1})^\top = (\hat{b}_{r,j})_{1 \leq r,j \leq m}$ ,  $\Delta c_{r,j} = \check{c}_{r,j}^{(\dot{s})} - \tilde{c}_{r,j}^{(\dot{s})}$  and  $\Delta \mathbf{f}_{r,j} = \ddot{\mathbf{f}}_{r,j}^{(\dot{s})} - \dot{\mathbf{f}}_{r,j}^{(\dot{s})}$  for  $1 \leq r, j \leq m$ . Applying (4.133), (4.135) and (4.141), we derive

$$\Delta c_{r,j} = \sum_{l=1}^m \Delta \mathbf{f}_{r,l} \hat{b}_{l,j} \quad \text{for } 1 \leq r, j \leq m. \quad (4.142)$$

From (4.134) and (4.139), we get

$$\Delta c_{r,j} = \check{c}_{r,j}^{(\dot{s})} - \tilde{c}_{r,j}^{(\dot{s})} = 0 \quad \text{for } r \in [(\dot{s} - 1)d_0\dot{m} + 1, m], \quad 1 \leq j \leq m. \quad (4.143)$$

By (4.139) and (4.132), we have

$$\tilde{c}_{r,j}^{(\dot{s})} = \sum_{l=1}^m \dot{\mathbf{f}}_{r,l}^{(\dot{s})} \hat{b}_{l,j} = \hat{b}_{r,j} \quad \text{for } r \in [(\dot{s} - 1)d_0\dot{m} + 1, m] \quad \text{and} \quad 1 \leq j \leq m. \quad (4.144)$$

Using (4.129), we obtain  $d_1^{(\dot{s})} > (\dot{s} - 1)d_0\dot{m}$ . By (4.134), (4.142) and (4.144), we get

$$\Delta c_{r,j} = \sum_{l=d_1^{(\dot{s})}}^{d_2^{(\dot{s})}} \Delta \mathbf{f}_{r,l} \tilde{c}_{l,j} \quad \text{for } r \in [1, (\dot{s} - 1)d_0\dot{m}] \quad \text{and} \quad 1 \leq j \leq m. \quad (4.145)$$

**Lemma 20.** *With notations as above. Let  $\dot{s} \geq 3$ ,  $(\tilde{\mathbf{x}}_n)_{0 \leq n < b^m}$  be a digital  $(t, m, \dot{s})$  net in base  $b$ ,  $\tilde{x}_n^{\dot{s}} \neq \tilde{x}_k^{\dot{s}}$  for  $n \neq k$ . Then  $(\check{\mathbf{x}}_n)_{0 \leq n < b^m}$  is a digital  $(t, m, \dot{s})$  net in base  $b$  with  $\check{x}_n^{\dot{s}} \neq \check{x}_k^{\dot{s}}$  for  $n \neq k$ ,*

$$\|\check{\mathbf{x}}_n^{(\dot{s})}\|_b = \|\tilde{\mathbf{x}}_n^{(\dot{s})}\|_b \quad \text{for } 0 < n < b^m \quad (4.146)$$

and

$$\Lambda = \mathbb{F}_b^{\dot{s}d_0\dot{m}}, \quad \text{for } m \geq 2d_0\dot{s}, \quad \dot{m} = [(m-t)/(2d_0(\dot{s}-1))], \quad (4.147)$$

where

$$\Lambda = \{(\check{y}_{n,d_1^{(1)}}^{(1)}, \dots, \check{y}_{n,d_2^{(1)}}^{(1)}, \dots, \check{y}_{n,d_1^{(\dot{s})}}^{(\dot{s})}, \dots, \check{y}_{n,d_2^{(\dot{s})}}^{(\dot{s})}) \mid n \in [0, b^m)\}$$

with  $d_1^{(i)} = 1$ ,  $d_2^{(i)} = d_0\dot{m}$  for  $1 \leq i < \dot{s}$ ,  $d_1^{(\dot{s})} = m - t + 1 - (\dot{s} - 1)d_0\dot{m}$  and  $d_2^{(\dot{s})} = m - t - (\dot{s} - 2)d_0\dot{m}$ .

**Proof.** By (4.140), we have  $\check{\mathbf{y}}_n = \check{\mathbf{y}}_{n'}$ ,  $\check{\mathbf{x}}_n = \check{\mathbf{x}}_{n'}$  and  $\check{\mathbf{y}}_n = \dot{\mathbf{y}}_{n'}$ ,  $\check{\mathbf{x}}_n = \dot{\mathbf{x}}_{n'}$  for  $\mathbf{n}' = \mathbf{n}B^{-1}$ . Hence, in order to prove the lemma, it is sufficient to take  $\check{\mathbf{x}}_n$  instead of  $\check{\mathbf{x}}_n$  and  $\dot{\mathbf{x}}_n$  instead of  $\check{\mathbf{x}}_n$ . Applying (4.137) and (4.138), we derive that  $\ddot{x}_n^{\dot{s}} \neq \ddot{x}_k^{\dot{s}}$  for  $n \neq k$ .

Suppose that  $a_j(n) = 0$  for  $1 \leq j \leq (\dot{s} - 1)d_0\dot{m}$ . By (4.134) and (4.136), we get  $\|\ddot{x}_n^{\dot{s}}\|_b = \|\dot{x}_n^{\dot{s}}\|_b$ .

Let  $a_j(n) = 0$  for  $1 \leq j < j_0 \leq (\dot{s} - 1)d_0\dot{m}$  and let  $a_{j_0}(n) \neq 0$ . From (4.134) and (4.136), we have  $\|\ddot{x}_n^{(\dot{s})}\|_b = \|\dot{x}_n^{(\dot{s})}\|_b = b^{-j_0}$ . Hence  $\|\check{\mathbf{x}}_n^{(\dot{s})}\|_b = \|\dot{\mathbf{x}}_n^{(\dot{s})}\|_b$  for all  $n \in [1, b^m)$  and (4.146) follows.

Let  $\mathbf{d} = (d_1, \dots, d_{\dot{s}})$ ,  $d_i \geq 0$  ( $i = 1, \dots, \dot{s}$ ),  $\check{\mathbf{v}}_{\mathbf{d}} = (\ddot{v}_1^{(1)}, \dots, \ddot{v}_{d_1}^{(1)}, \dots, \ddot{v}_1^{(\dot{s})}, \dots, \ddot{v}_{d_{\dot{s}}}^{(\dot{s})}) \in \mathbb{F}_b^{\dot{d}}$ , with  $\dot{d} = d_1 + \dots + d_{\dot{s}}$ , and let

$$\ddot{U}_{\check{\mathbf{v}}_{\mathbf{d}}} = \{0 \leq n < b^m \mid \ddot{y}_{n,j}^{(i)} = v_j^{(i)}, \quad 1 \leq j \leq d_i, \quad 1 \leq i \leq \dot{s}\}. \quad (4.148)$$

In order to prove that  $(\check{\mathbf{x}}_n)_{0 \leq n < b^m}$  is a  $(t, m, \dot{s})$  net, it is sufficient to verify that  $\#\ddot{U}_{\check{\mathbf{v}}_{\mathbf{d}}} = b^{m-\dot{d}}$  for all  $\check{\mathbf{v}}_{\mathbf{d}} \in \mathbb{F}_b^{\dot{d}}$  and all  $\mathbf{d}$  with  $\dot{d} \leq m - t$ . By (4.133), (4.134) and (4.135), we get

$$\dot{\mathbf{y}}_n^{(i)} = \sum_{j=1}^m \bar{a}_j(n) \dot{\mathbf{f}}_j^{(i)} \quad \text{and} \quad \ddot{\mathbf{y}}_n^{(i)} = \sum_{j=1}^m \bar{a}_j(n) \ddot{\mathbf{f}}_j^{(i)}, \quad \text{with} \quad \ddot{\mathbf{f}}_j^{(i)} = \dot{\mathbf{f}}_j^{(i)} \quad (4.149)$$

for  $1 \leq i \leq \dot{s} - 1$ ,  $1 \leq j \leq m$  and  $i = \dot{s}$ ,  $(\dot{s} - 1)d_0\dot{m} + 1 \leq j \leq m$ ,  $0 \leq n < b^m$ . Hence

$$\dot{\mathbf{y}}_n^{(i)} - \ddot{\mathbf{y}}_n^{(i)} = 0 \quad \text{for } 1 \leq i \leq \dot{s} - 1, \quad \dot{\mathbf{y}}_n^{(\dot{s})} - \ddot{\mathbf{y}}_n^{(\dot{s})} = \sum_{r=1}^{(\dot{s}-1)d_0\dot{m}} \bar{a}_r(n) (\dot{\mathbf{f}}_r^{(\dot{s})} - \ddot{\mathbf{f}}_r^{(\dot{s})}) \quad (4.150)$$

and  $\dot{\mathbf{y}}_{n,j}^{(\dot{s})} - \ddot{\mathbf{y}}_{n,j}^{(\dot{s})} = 0$  for  $j \in [1, (\dot{s} - 1)d_0\dot{m}]$ ,  $0 \leq n < b^m$ . Let

$\dot{v}_j^{(i)} := \ddot{v}_j^{(i)}$  for  $j \in [1, d_i]$ ,  $i \in [1, \dot{s} - 1]$  and  $\dot{v}_j^{(\dot{s})} := \ddot{v}_j^{(\dot{s})}$  for  $j \in [1, \min(d_{\dot{s}}, (\dot{s} - 1)d_0\dot{m})]$ .

For  $d_{\dot{s}} > (\dot{s} - 1)d_0\dot{m}$  and  $j \in [(\dot{s} - 1)d_0\dot{m} + 1, d_{\dot{s}}]$ , we define

$$\dot{v}_j^{(\dot{s})} = \ddot{v}_j^{(\dot{s})} + \sum_{r=1}^{(\dot{s}-1)d_0\dot{m}} \ddot{v}_r^{(\dot{s})}(\dot{\mathfrak{f}}_{r,j}^{(\dot{s})} - \ddot{\mathfrak{f}}_{r,j}^{(\dot{s})}).$$

By (4.132) and (4.149), we get

$$\dot{y}_{n,j}^{(\dot{s})} = \dot{v}_j^{(\dot{s})} \iff \bar{a}_j(n) = \dot{v}_j^{(\dot{s})} = \ddot{v}_j^{(\dot{s})}, \quad \text{for } j \in [1, \min(d_{\dot{s}}, (\dot{s} - 1)d_0\dot{m})], \quad n \in [0, b^m).$$

Using (4.150), we obtain for  $n \in [0, b^m)$  that

$$\ddot{\mathbf{y}}_{n,j}^{(i)} = \ddot{v}_j^{(i)} \iff \dot{\mathbf{y}}_{n,j}^{(i)} = \dot{v}_j^{(i)} \quad \text{for } 1 \leq j \leq d_i, \quad 1 \leq i \leq \dot{s}. \quad (4.151)$$

Let

$$\dot{U}_{\dot{\mathbf{v}}_{\mathbf{d}}} = \{0 \leq n < b^m \mid \dot{y}_{n,j}^{(i)} = \dot{v}_j^{(i)}, \quad 1 \leq j \leq d_i, \quad 1 \leq i \leq \dot{s}\}$$

with  $\dot{\mathbf{v}}_{\mathbf{d}} = (\dot{v}_1^{(1)}, \dots, \dot{v}_{d_1}^{(1)}, \dots, \dot{v}_1^{(\dot{s})}, \dots, \dot{v}_{d_{\dot{s}}}^{(\dot{s})})$ .

Taking into account that  $(\dot{\mathbf{x}}_n)_{0 \leq n < b^m}$  is a  $(t, m, \dot{s})$  net in base  $b$ , we get from (4.148) and (4.151) that  $\#\ddot{U}_{\dot{\mathbf{v}}_{\mathbf{d}}} = \#\dot{U}_{\dot{\mathbf{v}}_{\mathbf{d}}} = b^{m-d}$ .

Now consider the statement (4.147). Let  $\ddot{\mathbf{v}} = (\ddot{v}_{d_1^{(1)}}^{(1)}, \dots, \ddot{v}_{d_2^{(2)}}^{(1)}, \dots, \ddot{v}_{d_1^{(\dot{s})}}^{(\dot{s})}, \dots, \ddot{v}_{d_2^{(\dot{s})}}^{(\dot{s})}) \in \mathbb{F}_b^{\dot{d}}$ , with  $\dot{d} = d_2^{(1)} + \dots + d_2^{(\dot{s}-1)} + d_2^{(\dot{s})} - d_1^{(\dot{s})} + 1$ . It is easy to see that to obtain (4.147), it is sufficient to verify that  $\ddot{U}'_{\ddot{\mathbf{v}}} \neq \emptyset$  for all  $\ddot{\mathbf{v}} \in \mathbb{F}_b^{\dot{d}}$ , where

$$\ddot{U}'_{\ddot{\mathbf{v}}} = \{0 \leq n < b^m \mid \ddot{y}_j^{(i)} = \ddot{v}_j^{(i)}, \quad d_1^{(i)} \leq j \leq d_2^{(i)}, \quad 1 \leq i \leq \dot{s}\}.$$

According to (4.135) and (4.136),  $\ddot{U}'_{\ddot{\mathbf{v}}} \neq \emptyset$  if there exists  $n \in [0, b^m)$  such that

$$\sum_{r=1}^m \bar{a}_r(n) \ddot{\mathfrak{f}}_{j,r}^{(i)} = \ddot{v}_j^{(i)} \quad \text{for all } d_1^{(i)} \leq j \leq d_2^{(i)} \quad \text{and } 1 \leq i \leq \dot{s}. \quad (4.152)$$

By (4.132) and (4.134), we have that (4.152) is true only if  $\bar{a}_j(n) = \ddot{v}_j^{(\dot{s})}$

for  $d_1^{(\dot{s})} \leq j \leq d_2^{(\dot{s})}$ . Let  $n_0 = \sum_{j=d_1^{(\dot{s})}}^{d_2^{(\dot{s})}} \phi^{-1}(\ddot{v}_j^{(\dot{s})})b^{j-1}$  and let

$$n = n_0 + \sum_{i=1}^{\dot{s}-1} \sum_{j=d_1^{(i)}}^{d_2^{(i)}} \phi(\ddot{v}_j^{(i)} - \ddot{y}_{n_0,j}^{(i)})b^{(i-1)d_0\dot{m}+j-1}.$$



Therefore  $\bar{a}_j(n) = \ddot{v}_j^{(\dot{s})}$  for  $j \in [d_1^{(\dot{s})}, d_2^{(\dot{s})}]$  and  $\bar{a}_{(i-1)d_0m+j}(n) = \ddot{v}_j^{(i)}$  for  $j \in [d_1^{(i)}, d_2^{(i)}]$ ,  $i \in [1, \dot{s} - 1]$ . Using (4.132) and (4.134), we get that (4.152) is true and  $\dot{U}_{\dot{\mathbf{v}}} \neq \emptyset$  for all  $\dot{\mathbf{v}} \in \mathbb{F}_b^{\dot{d}}$ . Hence (4.147) is proved, and Lemma 20 follows.  $\square$

**End of the proof of Theorem 6.** Let  $C^{(1)}, \dots, C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$  be the generating matrices of a digital  $(t, s)$  sequence  $(\mathbf{x}_n)_{n \geq 0}$ . For any  $m \in \mathbb{N}$  we denote the  $m \times m$  left-upper sub-matrix of  $C^{(i)}$  by  $[C^{(i)}]_m$ .

Let  $m_k = s^2 d_0 (2^{2k+2} - 1)$ ,  $k = 0, 1, \dots$ ,

$$x_n^{(i,k)} = \sum_{j=1}^{m_k} \phi^{-1}(y_{n,j}^{(i,k)})/b^j, \quad \mathbf{y}_n^{(i,k)} = \mathbf{n}[C^{(i)}]_{m_k}^{\top} \quad (4.153)$$

and  $\mathbf{y}_n^{(i,k)} = (y_{n,1}^{(i,k)}, \dots, y_{n,m_k}^{(i,k)})$  for  $n \in [0, b^{m_k}]$ ,  $i \in [1, s]$ .

For  $x = \sum_{j \geq 1} x_j p_i^{-j}$ , where  $x_i \in Z_b = \{0, \dots, b-1\}$ , we define the truncation

$$[x]_m = \sum_{1 \leq j \leq m} x_j b^{-j} \quad \text{with } m \geq 1.$$

If  $x = (x^{(1)}, \dots, x^{(s)}) \in [0, 1]^s$ , then the truncation  $[\mathbf{x}]_m$  is defined coordinatewise, that is,  $[\mathbf{x}]_m = ([x^{(1)}]_m, \dots, [x^{(s)}]_m)$ .

By (2.14) - (2.16), we have

$$[\mathbf{x}_n]_{m_k} = \mathbf{x}_n^{(k)} := (x_n^{(1,k)}, \dots, x_n^{(s,k)}) \quad \text{for } n \in [0, b^{m_k}]. \quad (4.154)$$

Let  $\hat{C}^{(s+1,0)} = (\hat{c}_{i,j}^{(s+1,0)})_{1 \leq i,j \leq m_0}$  with  $\hat{c}_{i,j}^{(s+1,0)} = \delta_{i,m_0-j+1}$ ,  $i, j = 1, \dots, m_0$ . We will use (4.127) - (4.141) to construct a sequence of matrices  $\hat{C}^{(s+1,k)} \in \mathbb{F}_b^{m_k \times m_k}$  ( $k = 1, 2, \dots$ ), satisfying the following induction assumption:

For given sequence of matrices  $\hat{C}^{(s+1,0)}, \dots, \hat{C}^{(s+1,k-1)}$  there exists a matrix  $\hat{C}^{(s+1,k)} = (\hat{c}_{i,j}^{(s+1,k)})_{1 \leq i,j \leq m_k}$  such that

$$\hat{c}_{m_k-i+1,j}^{(s+1,k)} = \hat{c}_{m_{k-1}-i+1,j}^{(s+1,k-1)} \quad \text{for } i, j \in [1, m_{k-1}] \quad \text{and} \quad \hat{c}_{m_k-i+1,j}^{(s+1,k)} = 0 \quad (4.155)$$

for  $i \in [m_{k-1} + 1, m_k]$ ,  $j \in [1, m_{k-1}]$ ,  $(x_n^{(1,k)}, \dots, x_n^{(s,k)}, \hat{x}_n^{(s+1,k)})_{0 \leq n < b^{m_k}}$  is a  $(t, m_k, s+1)$  net in base  $b$  with

$$\hat{x}_n^{(s+1,k)} \neq \hat{x}_l^{(s+1,k)} \quad \text{for } n \neq l \quad \text{and} \quad \|\hat{x}_n^{(s+1,k)}\|_b = \|n\|_b b^{-m_k} \quad \text{for } 0 \leq n < b^{m_k}, \quad (4.156)$$

where

$$\hat{x}_n^{(s+1,k)} = \sum_{j=1}^{m_k} \phi^{-1}(y_{n,j}^{(s+1,k)})/b^j, \quad \mathbf{y}_n^{(s+1,k)} = \mathbf{n}\hat{C}^{(s+1,m_k)}{}^\top \quad (4.157)$$

and  $\mathbf{y}_n^{(s+1,k)} = (y_{n,1}^{(s+1,k)}, \dots, y_{n,m_k}^{(s+1,k)})$  for  $n \in [0, b^{m_k})$ .

Let  $k = 1$ . We take  $\hat{c}_{i,j}^{(s+1,1)} = \delta_{i,m_1-j+1}$  for  $i, j = 1, \dots, m_1$ . Now assume we know  $\hat{C}^{(s+1,k)}$  and we want to construct  $\hat{C}^{(s+1,k+1)}$ . We first construct  $\tilde{C}^{(s+1,k+1)} = (\tilde{c}_{i,j}^{(s+1,k+1)})_{1 \leq i,j \leq m_{k+1}}$  as following

$$\tilde{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)} = \hat{c}_{m_k-i+1,j}^{(s+1,k)} \quad \text{for } i, j \in [1, m_k], \quad \tilde{c}_{i,j}^{(s+1,k+1)} = \delta_{i,m_{k+1}-j+1} \quad (4.158)$$

$$\text{for } i \in [1, m_{k+1} - m_k], j \in [1, m_{k+1}] \quad \text{and} \quad \tilde{c}_{i,j}^{(s+1,k+1)} = \bar{0}$$

for  $(i, j) \in [1, m_{k+1} - m_k] \times [1, m_k]$  and  $(i, j) \in [m_{k+1} - m_k + 1, m_{k+1}] \times [m_k + 1, m_{k+1}]$ .

**Lemma 21.** *With notations as above,  $(x_n^{(1,k+1)}, \dots, x_n^{(s,k+1)}, \tilde{x}_n^{(s+1,k+1)})_{0 \leq n < b^{m_{k+1}}}$  is a  $(t, m_{k+1}, s+1)$  net in base  $b$  with  $\tilde{x}_n^{(s+1,k+1)} \neq \tilde{x}_l^{(s+1,k+1)}$  for  $n \neq l$ , and*

$$\|\tilde{x}_n^{(s+1,k+1)}\|_b = \|n\|_b b^{-m_{k+1}} \quad \text{for } 0 < n < b^{m_{k+1}}. \quad (4.159)$$

**Proof.** Let  $\mathbf{d} = (d_1, \dots, d_{s+1})$ ,  $\mathbf{v}_\mathbf{d} = (v_1^{(1)}, \dots, v_{d_1}^{(1)}, \dots, v_1^{(s+1)}, \dots, v_{d_{s+1}}^{(s+1)}) \in \mathbb{F}_b^{\mathbf{d}}$  with  $\dot{d} = d_1 + \dots + d_{s+1}$ ,

$$\begin{aligned} \tilde{U}_{\mathbf{v}_\mathbf{d}} &= \{0 \leq n < b^{m_{k+1}} \mid y_{n,j}^{(i,k)} = v_j^{(i)}, \quad 1 \leq j \leq d_i, \quad 1 \leq i \leq s \\ &\quad \text{and} \quad \tilde{y}_{n,j}^{(s+1,k+1)} = v_j^{(s+1)}, \quad 1 \leq j \leq d_{s+1}\}. \end{aligned} \quad (4.160)$$

In order to prove that  $(x_n^{(1,k+1)}, \dots, x_n^{(s,k+1)}, \tilde{x}_n^{(s+1,k+1)})_{0 \leq n < b^{m_{k+1}}}$  is a  $(t, m_{k+1}, s+1)$  net, it is sufficient to verify that  $\#\tilde{U}_{\mathbf{v}_\mathbf{d}} = b^{m_{k+1}-\dot{d}}$  for all  $\mathbf{v}_\mathbf{d} \in \mathbb{F}_b^{\mathbf{d}}$  and all  $\mathbf{d}$  with  $\dot{d} \leq m_{k+1} - t$ .

Suppose that  $d_{s+1} \leq m_{k+1} - m_k$ . Let  $n \in [0, b^{m_{k+1}})$ ,  $n_0 \equiv n \pmod{b^{m_{k+1}-d_{s+1}}}$ ,  $n_0 \in [0, b^{m_{k+1}-d_{s+1}})$  and let  $n_1 = n - n_0$ . It is easy to see that

$$\tilde{y}_{n,j}^{(s+1,k+1)} = \tilde{y}_{n_0,j}^{(s+1,k+1)} + \tilde{y}_{n_1,j}^{(s+1,k+1)}.$$

Let  $j \in [1, m_{k+1} - m_k]$ . By (4.158), we get

$$\tilde{y}_{n,j}^{(s+1,k+1)} = \sum_{r=1}^{m_{k+1}} \bar{a}_r(n) \tilde{c}_{j,r}^{(s+1,k+1)} = \sum_{r=1}^{m_{k+1}-m_k} \bar{a}_r(n) \delta_{j,m_{k+1}+1-r} = \bar{a}_{m_{k+1}+1-j}(n). \quad (4.161)$$

Let  $\ddot{n} = \sum_{j=1}^{d_{s+1}} \phi(v_j^{(s+1)}) b^{m_{k+1}-j}$ . By (4.160), we get  $n \in \tilde{U}_{\mathbf{v}_d} \Leftrightarrow n_1 = \ddot{n}$  and  $n_0 \in \tilde{U}'_{\mathbf{v}_d}$ , where

$$\tilde{U}'_{\mathbf{v}_d} = \{0 \leq \dot{n} < b^{m_{k+1}-d_{s+1}} \mid y_{\dot{n},j}^{(i,k+1)} = v_j^{(i)} - y_{\dot{n},j}^{(i,k+1)}, 1 \leq j \in [1, d_i], i \in [1, s]\}.$$

Bearing in mind (4.157), (4.158), (4.160) and that  $(\mathbf{x}(n))_{0 \leq n < b^{m_{k+1}-d_{s+1}}}$  is a  $(t, m_{k+1} - d_{s+1}, s)$  net in base  $b$ , we obtain  $\#\tilde{U}_{\mathbf{v}_d} = \#\tilde{U}'_{\mathbf{v}_d} = b^{m_{k+1}-d}$ .

Now let  $d_{s+1} > m_{k+1} - m_k$ . Let  $n \in [0, b^{m_{k+1}})$ ,  $n_0 \equiv n \pmod{b^{m_k}}$ ,  $n_0 \in [0, b^{m_k})$  and let  $n_1 = n - n_0$ . We have

$$\tilde{y}_{n,j}^{(s+1,k+1)} = \tilde{y}_{n_0,j}^{(s+1,k+1)} + \tilde{y}_{n_1,j}^{(s+1,k+1)}.$$

Let  $\ddot{n} = \sum_{j=1}^{m_{k+1}-m_k} \phi(v_j^{(s+1)}) b^{m_{k+1}-j}$ . By (4.160) and (4.161), we get

$$\begin{aligned} n \in \tilde{U}_{\mathbf{v}_d} \Leftrightarrow n_1 = \ddot{n} \text{ and } n_0 \in \{0 \leq \dot{n} < b^{m_k} \mid y_{\dot{n},j}^{(i,k+1)} = v_j^{(i)} - y_{\dot{n},j}^{(i,k+1)}, 1 \leq j \leq d_i, \\ 1 \leq i \leq s \text{ and } y_{\dot{n},j}^{(s+1,k+1)} = v_j^{(s+1)} - y_{\dot{n},j}^{(s+1,k+1)}, m_{k+1} - m_k + 1 \leq j \leq d_{s+1}\}. \end{aligned}$$

Let  $j \in [m_{k+1} - m_k + 1, m_{k+1}]$  and let  $j_0 = m_{k+1} + 1 - j \in [1, m_k]$ . By (4.158), we derive

$$\begin{aligned} \tilde{y}_{n,j}^{(s+1,k+1)} &= \tilde{y}_{\dot{n},m_{k+1}+1-j_0}^{(s+1,k+1)} = \sum_{r=1}^{m_{k+1}} \bar{a}_r(\dot{n}) \tilde{c}_{m_{k+1}+1-j_0,r}^{(s+1,k+1)} = \sum_{r=1}^{m_k} \bar{a}_r(\dot{n}) \tilde{c}_{m_{k+1}+1-j_0,r}^{(s+1,k+1)} \\ &= \sum_{r=1}^{m_k} \bar{a}_r(\dot{n}) \tilde{c}_{m_k+1-j_0,r}^{(s+1,k)} = \tilde{y}_{\dot{n},m_k+1-j_0}^{(s+1,k)} \text{ for all } \dot{n} \in [0, b^{m_k}). \end{aligned} \quad (4.162)$$

We have that  $y_{\dot{n},j}^{(i,k+1)} = y_{\dot{n},j}^{(i,k)}$  ( $i = 1, \dots, s$ ) and  $y_{\dot{n},j}^{(s+1,k+1)} = y_{\dot{n},m_k+1-j_0}^{(s+1,k)}$  for  $\dot{n} \in [0, b^{m_k})$ . Hence

$$n \in \tilde{U}_{\mathbf{v}_d} \Leftrightarrow n_1 = \ddot{n} \text{ and } n_0 \in \tilde{U}'_{\mathbf{v}_d} = \left\{0 \leq \dot{n} < b^{m_k} \mid y_{\dot{n},j}^{(i,k)} = v_j^{(i)} - y_{\dot{n},j}^{(i,k+1)}, j \in [1, d_i], \right.$$

$$i \in [1, s], \text{ and } y_{n, j-m_{k+1}+m_k}^{(s+1, k)} = v_{j-m_{k+1}+m_k}^{(s+1)} - y_{n, j}^{(s+1, k+1)}, \quad j \in (m_{k+1} - m_k, d_{s+1}].$$

Taking into account that  $(x_n^{(1, k)}, \dots, x_n^{(s, k)}, \tilde{x}_n^{(s+1, k)})_{0 \leq n < b^{m_k}}$  is a  $(t, m_k, s+1)$  net in base  $b$ , we obtain  $\#\tilde{U}_{\mathbf{v}_d} = \#\tilde{U}'_{\mathbf{v}_d} = b^{m_k - (d - m_{k+1} + m_k)} = b^{m_{k+1} - d}$ . Therefore  $(x_n^{(1, k+1)}, \dots, x_n^{(s, k+1)}, \tilde{x}_n^{(s+1, k+1)})_{0 \leq n < b^{m_{k+1}}}$  is a  $(t, m_{k+1}, s+1)$  net in base  $b$ .

From (4.158), (4.161), (4.162) and the induction assumption, we get that  $\tilde{x}_n^{(s+1, k+1)} \neq \tilde{x}_l^{(s+1, k+1)}$  for  $n \neq l$ .

Consider the assertion (4.159). Let  $n \in [0, b^{m_{k+1}})$  and let

$$\|\tilde{x}_n^{(s+1, k+1)}\|_b = b^{-j_1}. \quad (4.163)$$

Hence  $\tilde{y}_{n, j}^{(s+1, k+1)} = 0$  for  $1 \leq j \leq j_1 - 1$  and  $\tilde{y}_{n, j_1}^{(s+1, k+1)} \neq 0$  (see (1.4)).

Let  $j_1 \in [1, m_{k+1} - m_k]$ . By (4.161), we get  $\bar{a}_{m_{k+1}+1-j}(n) = 0$  for  $1 \leq j \leq j_1 - 1$  and  $\bar{a}_{m_{k+1}+1-j_1}(n) \neq 0$ . Therefore  $\|n\|_b = \left\| \sum_{i=1}^{m_{k+1}} a_i(n) b^{i-1} \right\|_b = b^{m_{k+1} - j_1}$ .

Now let  $j_1 \in [m_{k+1} - m_k + 1, m_{k+1}]$ . From (4.161), we obtain  $\bar{a}_{m_{k+1}+1-j}(n) = 0$  for  $1 \leq j \leq m_{k+1} - m_k$ . Hence  $n \in [0, b^{m_k})$ . Using (4.158) and (4.161), we have  $\tilde{y}_{n, j}^{(s+1, k)} = \tilde{y}_{n, j-m_{k+1}+m_k}^{(s+1, k)}$  for  $m_{k+1} - m_k + 1 \leq j \leq j_1$ . Therefore  $\tilde{y}_{n, j}^{(s+1, k)} = 0$  for  $1 \leq j \leq j_1 - m_{k+1} + m_k - 1$  and  $\tilde{y}_{n, j_1 - m_{k+1} + m_k}^{(s+1, k)} \neq 0$ . Using the induction assumption (4.156), we get  $b^{-j_1 + m_{k+1} - m_k} = \|\tilde{x}_n^{(s+1, k)}\|_b = \|n\|_b b^{-m_k}$ .

By (4.163), we obtain  $\|\tilde{x}_n^{(s+1, k+1)}\|_b = \|n\|_b b^{-m_{k+1}}$ . Thus assertion (4.159) is proved and Lemma 21 follows.  $\square$

Now we apply (4.127) - (4.141) with  $s = s+1$ ,  $m = m_{k+1}$ ,  $\tilde{C}^{(i)} := [C^{(i)}]_{m_{k+1}}$  ( $i = 1, \dots, s$ ) and  $\tilde{C}^{(s+1)} := \tilde{C}^{(s+1, k+1)}$  to construct matrices  $\check{C}^{(i)}$  ( $i = 1, \dots, s+1$ ).

From (4.141), we have

$$\check{C}^{(i)} = \tilde{C}^{(i)} = [C^{(i)}]_{m_{k+1}} \quad \text{for } i = 1, \dots, s. \quad (4.164)$$

Let  $\hat{C}^{(s+1, k+1)} := \check{C}^{(s+1)}$ . According to (4.143) and (4.158), we get

$$\hat{c}_{r, j}^{(s+1, k+1)} - \tilde{c}_{r, j}^{(s+1, k+1)} = 0 \quad \text{for } r \in [sd_0 m_{k+1} + 1, m_{k+1}] \text{ and } 1 \leq j \leq m_{k+1}. \quad (4.165)$$

By (4.129) and (4.145), we obtain for  $r \in [1, sd_0 m_{k+1}]$  and  $1 \leq j \leq m_{k+1}$

$$\hat{c}_{r, j}^{(s+1, k+1)} - \tilde{c}_{r, j}^{(s+1, k+1)} = \sum_{l=d_1^{(s+1, k+1)}}^{d_2^{(s+1, k+1)}} \Delta f_{r, l}^{(s+1, k+1)} \tilde{c}_{l, j}^{(s+1, k+1)}, \quad (4.166)$$

where  $d_1^{(s+1,k+1)} = m_{k+1} - t + 1 - sd_0\dot{m}_{k+1}$ ,  $d_2^{(s+1,k+1)} = m_{k+1} - t - (s-1)d_0\dot{m}_{k+1}$ ,  $m_{k+1} = s^2d_0(2^{2k+4} - 1)$ ,  $d_0 = d + t$  and  $\dot{m}_{k+1} = [(m_{k+1} - t)/(2sd_0)]$ . We have  $d_1^{(s+1,k+1)} > (s-1)d_0\dot{m}_{k+1}$ ,  $\dot{m}_{k+1} = 2^{2k+3} - 1$  for  $k = 0, 1, \dots$  and

$$m_{k+1} - d_2^{(s+1,k+1)} \geq (s-1)d_0\dot{m}_{k+1} \geq 2^{-1}s^2d_0(2^{2k+3} - 1) > m_k.$$

By (4.158), we obtain  $\tilde{c}_{r,j}^{(s+1,k+1)} = 0$  for  $r \leq d_2^{(s+1,k+1)} < m_{k+1} - m_k$  and  $1 \leq j \leq m_k$ . From (4.166), we derive

$$\hat{c}_{r,j}^{(s+1,k+1)} - \tilde{c}_{r,j}^{(s+1,k+1)} = 0 \quad \text{for } r \in [1, sd_0\dot{m}_{k+1}] \text{ and } 1 \leq j \leq m_k. \quad (4.167)$$

Bearing in mind that

$$m_{k+1} - sd_0\dot{m}_{k+1} = s^2d_0(2^{2k+4} - 1) - s^2d_0(2^{2k+3} - 1) = s^2d_02^{2k+3} > m_k,$$

we get from (4.165) and (4.158)

$$\hat{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)} = \tilde{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)} = \hat{c}_{m_k-i+1,j}^{(s+1,k)} \quad \text{for } 1 \leq i, j \leq m_k. \quad (4.168)$$

Applying (4.158), (4.165) and (4.167), we have

$$\hat{c}_{i,j}^{(s+1,k+1)} = \tilde{c}_{i,j}^{(s+1,k+1)} = 0, \text{ for } 1 \leq i \leq m_{k+1} - m_k, 1 \leq j \leq m_k$$

Now using (4.168), we obtain (4.155).

We see that (4.156) follows from (4.159) and (4.146). Consider the net  $(\hat{\mathbf{x}}_n^{(k+1)})_{n=0}^{b^{m_{k+1}}-1}$  with  $\hat{\mathbf{x}}_n^{(k+1)} = (x_n^{(1,k+1)}, \dots, x_n^{(s,k+1)}, \hat{x}_n^{(s+1,k+1)}) := \check{\mathbf{x}}_n = (\check{x}_n^{(1)}, \dots, \check{x}_n^{(s+1)})$ . Let

$$\Lambda_{k+1} = \left\{ \left( (y_{n,1}^{(i,k+1)}, \dots, y_{n,d(i,k+1)}^{(i,k+1)})_{1 \leq i \leq s}, \hat{y}_{n,d_1^{(s+1,k+1)}}^{(s+1,k+1)}, \dots, \hat{y}_{n,d_2^{(s+1,k+1)}}^{(s+1,k+1)} \right) \mid n \in [0, b^{m_{k+1}}) \right\}$$

with  $d^{(i,k+1)} = d_0\dot{m}_{k+1}$  for  $1 \leq i \leq s$ . Using (4.129), (4.164) and Lemma 20, we obtain

$$\Lambda_{k+1} = \mathbb{F}_b^{(s+1)d_0\dot{m}_{k+1}}, \quad \text{for } \dot{m}_{k+1} = [(m_{k+1} - t)/(2sd_0)] = s(2^{k+1} - 1), \quad (4.169)$$

and  $(\hat{\mathbf{x}}_n^{(k+1)})_{0 \leq n < b^{m_{k+1}}}$  is a  $(t, m_{k+1}, s+1)$  net in base  $b$ . Thus we have that  $\hat{C}^{(s+1,k+1)}$  satisfy the induction assumption.

Let  $C^{(s+1,k+1)} = (c_{i,j}^{(s+1,k+1)})_{1 \leq i,j \leq m_{k+1}}$  where  $c_{i,j}^{(s+1,k+1)} := \hat{c}_{m_{k+1}-i+1,j}^{(s+1,k+1)}$  for  $1 \leq i, j \leq m_{k+1}$ . By (4.155), we get

$$[C^{(s+1,k+1)}]_{m_k} = C^{(s+1,k)} \text{ and } c_{i,j}^{(s+1,k+1)} = 0, \quad i \in (m_k, m_{k+1}], j \in [1, m_k]. \quad (4.170)$$

Now let  $C^{(s+1)} = (c_{i,j}^{(s+1)})_{i,j \geq 1} = \lim_{k \rightarrow \infty} C^{(s+1,k)}$  i.e.  $[C^{(s+1)}]_{m_k} := C^{(s+1,k)}$ ,  $k = 1, 2, \dots$ . We define

$$h_k(n) := h_{k,1}(n) + \dots + h_{k,m_k}(n)b^{m_k-1} := \hat{x}_n^{(s+1,k)}b^{m_k} \quad \text{for } 0 \leq n < b^{m_k}. \quad (4.171)$$

From (4.157), we have

$$\begin{aligned} \phi(h_{k,i}(n)) &= \phi(\hat{x}_{n,m_k-i+1}^{(s+1,k)}) = \hat{y}_{n,m_k-i+1}^{(s+1,k)} = \sum_{j=1}^{m_k} \bar{a}_j(n) \hat{c}_{m_k-i+1,j}^{(s+1,k)} \\ &= \sum_{j=1}^{m_k} \bar{a}_j(n) c_{m_k-i+1,j}^{(s+1,k)} \quad \text{for } 0 \leq n < b^{m_k}. \end{aligned} \quad (4.172)$$

Applying (4.170), we obtain for  $n \in [0, b^{m_k})$  that

$$h_{k,i}(n) = 0 \text{ for } i > m_k \text{ and } h_k(n) = h_{k-1}(n) \in [0, b^{m_{k-1}}) \text{ for } n \in [0, b^{m_{k-1}}). \quad (4.173)$$

For  $n \in [1, b^{m_k})$ , we get from (4.172) and (4.156) that

$$\|h_k(n)\|_b = \|n\|_b. \quad (4.174)$$

Let  $l \neq n \in [0, b^{m_k})$ . Using (4.156), we have  $(\hat{y}_{l,1}^{(s+1,k)}, \dots, \hat{y}_{l,m_k}^{(s+1,k)}) \neq (\hat{y}_{n,1}^{(s+1,k)}, \dots, \hat{y}_{n,m_k}^{(s+1,k)})$ . Hence  $(h_{k,1}(l), \dots, h_{k,m_k}(l)) \neq (h_{k,1}(n), \dots, h_{k,m_k}(n))$  and  $h_k(l) \neq h_k(n)$ .

Therefore  $h_k$  is a bijection from  $[0, b^{m_k})$  to  $[0, b^{m_k})$ . We define  $h_k^{-1}(n)$  such that  $h_k(h_k^{-1}(n)) = n$  for all  $n \in [0, b^{m_k})$ .

Let  $n \in [0, b^{m_k})$  and  $l = h_k^{-1}(n)$ , then  $l \in [0, b^{m_k})$  and  $h_{k+1}(l) = h_k(l) = n$ . Thus

$$h_{k+1}^{-1}(n) = h_k^{-1}(n) = l \quad \text{for } n \in [0, b^{m_k}). \quad (4.175)$$

Let  $h(n) = \lim_{k \rightarrow \infty} h_k(n)$ , and  $h^{-1}(n) = \lim_{k \rightarrow \infty} h_k^{-1}(n)$ .

Let  $n \in [0, b^{m_k})$  and let  $l = h_k^{-1}(n)$ . By (4.173) and (4.175), we get

$$h(n) = h_k(n) = l, \quad h^{-1}(l) = h_k^{-1}(l) = n, \quad \text{and} \quad h^{-1}(h(n)) = n.$$

Consider the  $d$ -admissible property of the sequence  $(\mathbf{x}_{h^{-1}(n)})_{n \geq 0}$ . It is sufficient to take  $k = 0$  in (1.4).

Let  $n \in [0, b^{m_k})$ . By (4.174), we have  $\|h(n)\|_b = \|h_k(n)\|_b = \|n\|_b$ . Taking into account Definition 5 and that  $(\mathbf{x}_n)_{n \geq 0}$  is a  $d$ -admissible sequence, we obtain

$$\|n\|_b \|\mathbf{x}_{h^{-1}(n)}\|_b = \|h(l)\|_b \|\mathbf{x}_l\|_b = \|l\|_b \|\mathbf{x}_l\|_b \geq b^{-d}, \quad \text{with } l = h^{-1}(n). \quad (4.176)$$

Hence  $(\mathbf{x}_{h^{-1}(n)})_{n \geq 0}$  is a  $d$ -admissible sequence.

By the induction assumption,  $([\mathbf{x}_n]_{m_k}, h_k(n)/b^{m_k})_{0 \leq n < b^{m_k}}$  is a  $(t, m_k, s+1)$  net in base  $b$  for  $k \geq 1$ . Hence  $(\mathbf{x}_n, h(n)/b^{m_k})_{0 \leq n < b^{m_k}}$  and  $(\mathbf{x}_{h^{-1}(n)}, n/b^{m_k})_{0 \leq n < b^{m_k}}$  are also  $(t, m_k, s+1)$  nets in base  $b$  for  $k \geq 1$ . By Lemma 1,  $(\mathbf{x}_{h^{-1}(n)})_{n \geq 0}$  is a  $(t, s)$  sequence in base  $b$ .

Let  $N \in [b^{m_k}, b^{m_{k+1}})$ . Applying Lemma B, we get

$$\begin{aligned} \sigma &:= 1 + \min_{0 \leq Q < b^{m_k}, \mathbf{w} \in E_{m_k}^s} \max_{1 \leq M \leq N} MD^*((\mathbf{x}_{h^{-1}(n \ominus Q)} \oplus \mathbf{w})_{0 \leq n < M}) \\ &\geq 1 + \min_{0 \leq Q < b^{m_k}, \mathbf{w} \in E_{m_k}^s} \max_{1 \leq M \leq b^{m_k}} MD^*((\mathbf{x}_{h^{-1}(n \ominus Q)} \oplus \mathbf{w})_{0 \leq n < M}) \\ &\geq \min_{0 \leq Q < b^{m_k}, \mathbf{w} \in E_{m_k}^s} b^{m_k} D^*((\mathbf{x}_{h^{-1}(n \ominus Q)} \oplus \mathbf{w}, n/b^{m_k})_{0 \leq n < b^{m_k}}) \\ &\geq \min_{0 \leq Q < b^{m_k}, \mathbf{w} \in E_{m_k}^s} b^{m_k} D^*((\mathbf{x}_l \oplus \mathbf{w}, h(l) \oplus Q/b^{m_k})_{0 \leq l < b^{m_k}}) \end{aligned}$$

where  $l = h^{-1}(n \ominus Q)$  and  $n = h(l) \oplus Q$ . Bearing in mind that  $h(n) = h_k(n)$  for  $0 \leq n < b^{m_k}$ , and that  $\hat{x}_n^{(s+1, k)} = h_k(n)/b^{m_k}$  for  $0 \leq n < b^{m_k}$ , we get

$$\sigma \geq \min_{0 \leq Q < b^{m_k}, \mathbf{w} \in E_{m_k}^s} b^{m_k} D^*((\mathbf{x}_n \oplus \mathbf{w}, \hat{x}_n^{(s+1, k)} \oplus (Q/b^{m_k}))_{0 \leq n < b^{m_k}}). \quad (4.177)$$

By (4.176) and (1.4), we obtain that  $(\mathbf{x}_n, h(n)/b^{m_k})_{0 \leq n < b^{m_k}}$  is a  $d$ -admissible net.

Applying (4.154) and the induction assumption, we get that  $(\mathbf{x}_n, h(n)/b^{m_k})_{0 \leq n < b^{m_k}}$  is a  $(t, m_k, s+1)$  net in base  $b$ . Let

$$\Lambda'_k = \left\{ \left( (y_{n,1}^{(i)}, \dots, y_{n,d(i,k)}^{(i)})_{1 \leq i \leq s}, \hat{y}_{n,d_1^{(s+1,k)}}^{(s+1,k)}, \dots, \hat{y}_{n,d_2^{(s+1,k)}}^{(s+1,k)} \right) \mid n \in [0, b^{m_k}] \right\}.$$

Using (4.153), (4.154) and (4.171), we obtain  $y_{n,j}^{(i)} = y_{n,j}^{(i,k)}$  for  $1 \leq j \leq m_k$ ,  $1 \leq i \leq s$ , and  $h(n)/b^{m_k} = \hat{x}_n^{(s+1,k)}$ . By (4.169), we have

$$\Lambda'_k = \Lambda_k = \mathbb{F}_b^{(s+1)d_0 \dot{m}}, \quad \text{for } \dot{m} = [(m_k - t)/(2sd_0)] = d_2^{(s+1,k)} - d_1^{(s+1,k)} + 1.$$

Now we apply Corollary 2 with  $\dot{s} = s+1$ ,  $\epsilon = (2sd_0)^{-1}$ ,  $\eta = \hat{e} = 1$ ,  $\tilde{r} = t$ ,  $m = m_k$ ,  $\tilde{m} = m - t$ ,  $\ddot{m}_{s+1} = d_1^{(s+1,k)} - 1$ ,  $B_i = \emptyset$  for  $i \in [1, s+1]$ , and  $B = 0$ . Taking into account (4.177), we get the assertion in Theorem 6.  $\square$

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